Génération de plans discrets par substitutions généralisées

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Arithmetic discrete planes

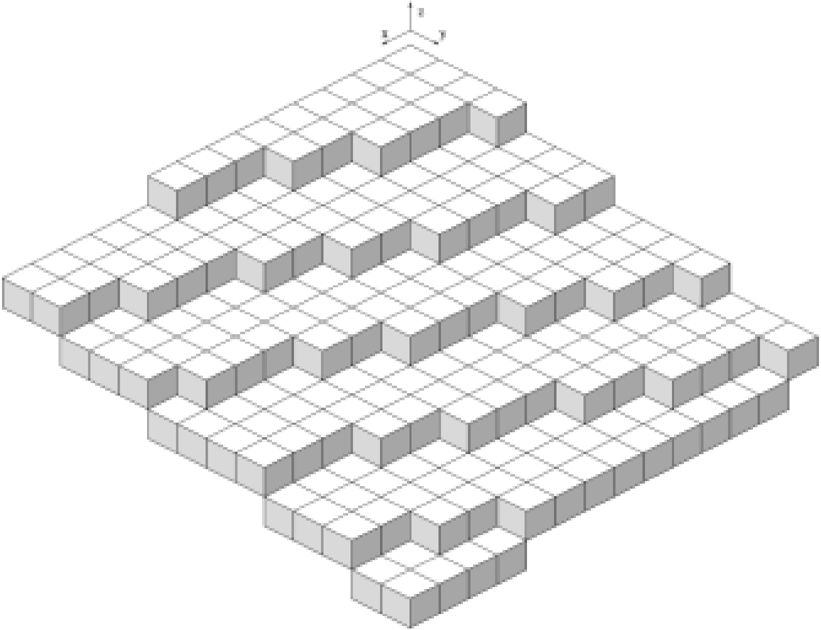
Definition (Réveillès, 1991)

The discrete plane $\mathcal{P}(a, b, c, \mu, \omega)$ is defined by

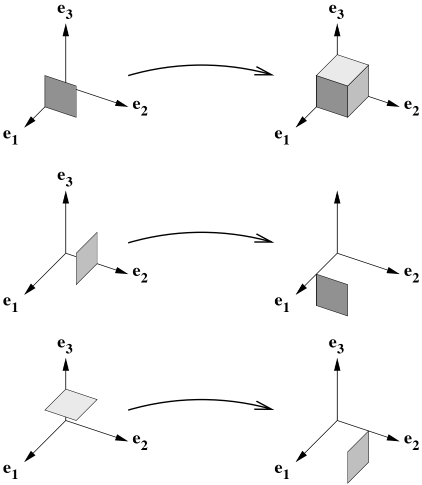
$$\mathcal{P}(a, b, c, \mu, \omega) = \left\{(x, y, z) \in \mathbb{Z}^3 \,|\, 0 \leq ax + by + cz + \mu < \omega\right\}.$$

- μ is the translation parameter of $\mathcal{P}(a, b, c, \mu, \omega)$,
- ω is the thickness of $\mathcal{P}(a, b, c, \mu, \omega)$.
- If $\omega = |a| + |b| + |c|$ then $\mathcal{P}(a, b, c, \mu, \omega)$ is said to be *standard*.

Discrete plane



Generalized substitutions



Droites discrètes et mots équilibrés

Arithmetic discrete lines

Definition (Reveillès, 1991)

The discrete line $\mathcal{L}(a, b, \mu, \omega)$ is defined by

$$\mathcal{L}(a, b, \mu, \omega) = \left\{(x, y) \in \mathbb{Z}^2 \,|\, 0 \leq ax + by + \mu < \omega\right\}.$$

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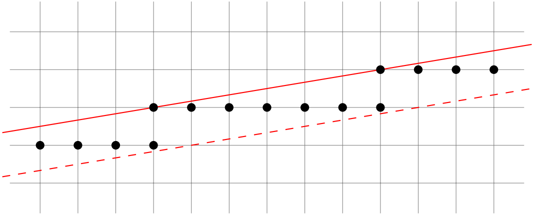

From discrete lines to words

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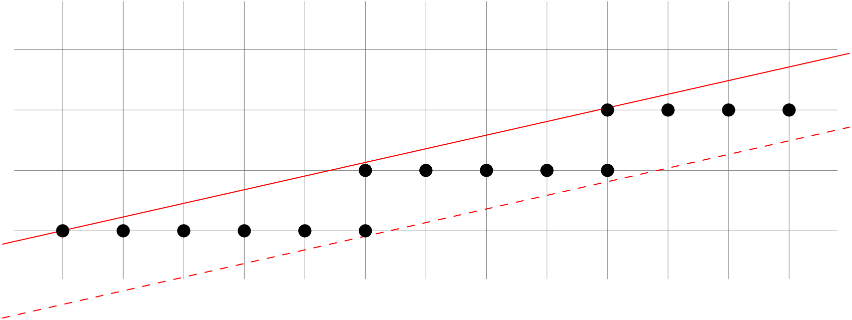

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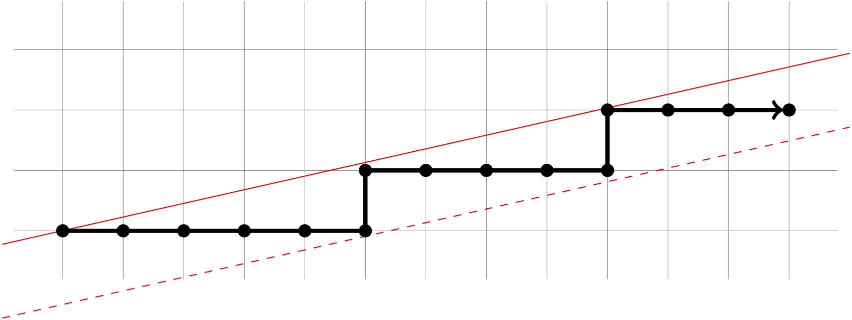

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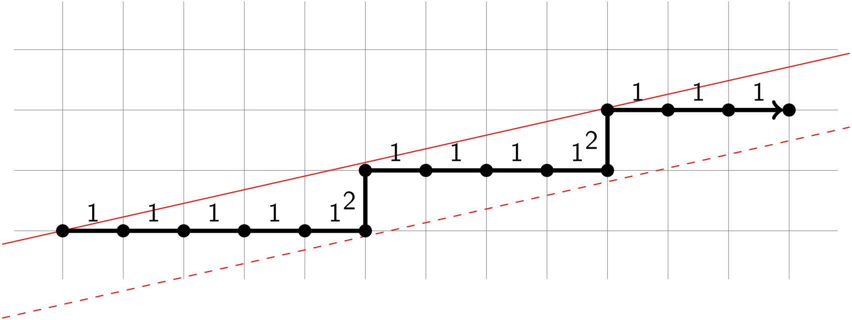

$$w = 1111121111 2111 \cdots$$

Sturmian words

Definition (Morse and Hedlund 1940)

An infinite word w over a two-letter alphabet is *Sturmian* if, equivalently,

- w admits exactly $n+1$ factors of length n,
- w is balanced and aperiodic,
- w codes (as in the previous slide) a standard discrete line with irrational slope $\alpha = b/a > 0$.

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Two Sturmian words u and v have same slope iff $\mathrm{Fact}(u) = \mathrm{Fact}(v)$.

Factors of Sturmian words

Definition

A finite word $u \in \{1, 2\}^*$ is called *central* if, equialently,

- u is *strictly bispecial*, that is $1u1, 1u2, 2u1$ and $2u2$ are all factors of Sturmian words.

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- u is either a single letter repeated or it is a palindrome and there exist two palindromes x and y such that $u = x12y$.

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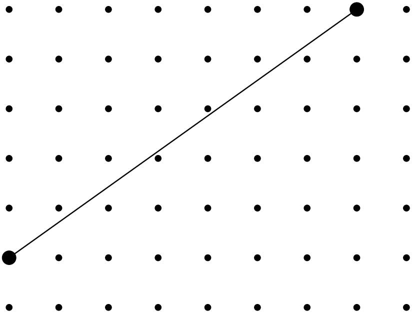
The set of Christoffel word is

$$w = \{1,2\} \cup \{1u2 | u \text{ is a central word}\}.$$

Geometrical description of Christoffel words



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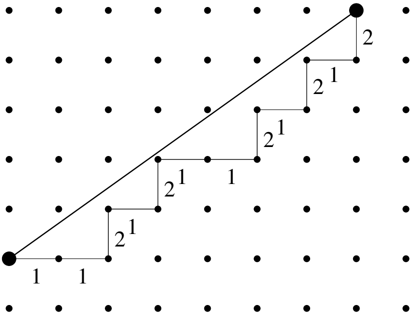

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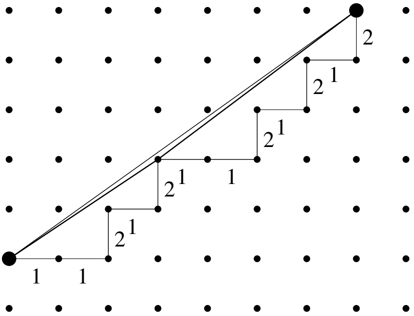
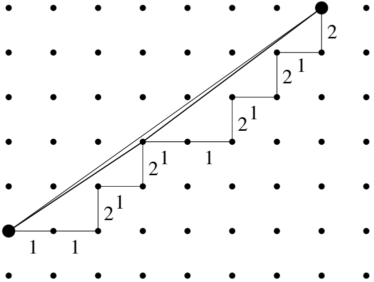


The *standard factorization* of a Christoffel word w is the only factorization $w = uv$ where u and v are both Christoffel words.

Geometrical description of Christoffel words



$$\begin{aligned}\alpha &= [z_0; z_1, z_2, \dots] \\ &= z_0 + \cfrac{1}{z_1 + \cfrac{1}{z_2 + \cfrac{1}{z_3 + \dots}}}\end{aligned}$$

The Christoffel word c_n of slope $[z_0; z_1, z_2, \dots, z_n]$ is given recursively by :

$$c_n = \begin{cases} c_{2m-2} c_{2m-1}^{z_{2m}} \text{ if } n = 2m, \\ c_{2m}^{z_{2m+1}} c_{2m-1} \text{ if } n = 2m+1. \end{cases} \qquad \text{where } c_{-1} = 2, \text{ and } c_{-2} = 1,$$

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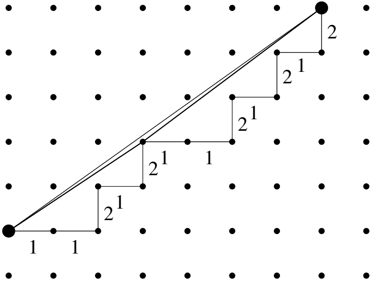

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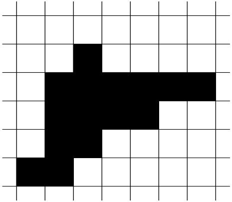
$\lim_{n \to \infty} s_n$ existe et est un mot Sturmien de pente α.

Digital convexity

Definition

A digital shape $S \subset \mathbb{Z}^2$ is *digitally convex* if its Euclidean convex hull E is such that $E \cap \mathbb{Z}^2 = S$.



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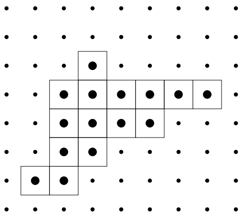

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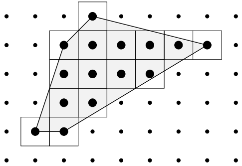

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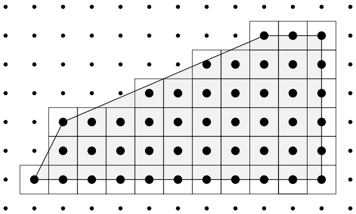

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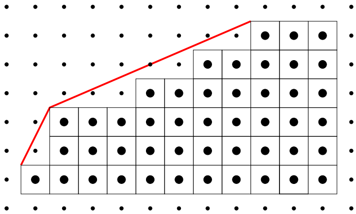

Theorem (Brlek, Lachaud, Reutenauer, P.)

A digital shape S is digitally convex if and only if each of its four quadrant words may be factorized as a sequence of decreasing Christoffel words. (If it exists, such a factorization is unique)

Factorization as decreasing Christoffel words

Proposition

If word $w \in \{1,2\}^*$ *is factorizes as a sequence of decreasing Christoffel words* $c_1^{n_1} \cdot c_2^{n_2} \cdots c_m^{n_m}$ *then* c_1 *is the longest prefix of* w *that is a Christoffel word.*

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Proposition

Three words u, v, w *with* $|u| < |v| \leq |w|$ *such that* u *and* v *are Christoffel words and both are prefixes of* w*, then there exist* $k \geq 1$ *such that*

$$u^k \cdot y \in Pref(v),$$

where $u = xy$ *is the standard factorization of* u*.*

Test de convexité

$$w = 1 \;\; 1 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \;\; 1 \;\; 1.$$

Test de convexité

$$w = \overset{\downarrow}{1}\ 1\ 1\ 2\ \underset{\uparrow}{1}\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1.$$

1 1 1 2 1 1 2

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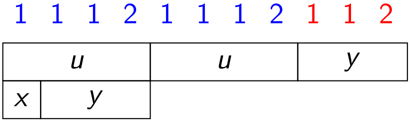
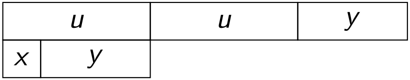


Test de convexité

$$w = 1\ \ 1\ \ 1\ \ \overset{\downarrow}{2}\ \ 1\ \ 1\ \ 1\ \ \underset{\uparrow}{2}\ \ 1\ \ 1\ \ 2\ \ 1\ \ 1\ \ 1\ \ 1\ \ 2\ \ 1\ \ 1\ \ 1\ \ 1.$$

$$\color{blue}{1\ \ 1\ \ 1\ \ 2\ \ 1\ \ 1\ \ 1\ \ 2}\ \ \color{red}{1\ \ 1\ \ 2}$$



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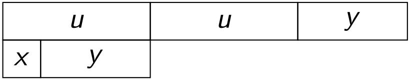

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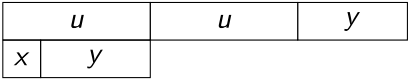

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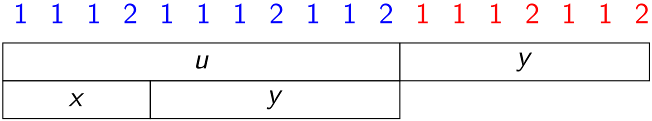

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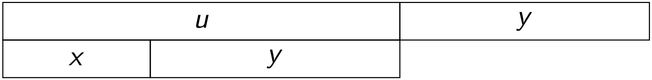

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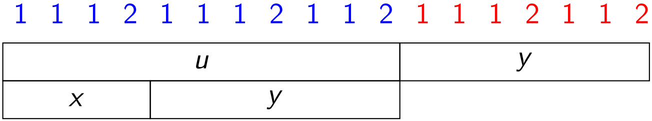

Test de convexité

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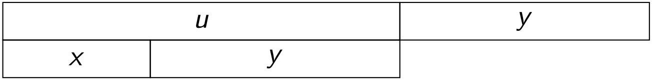

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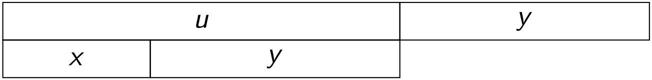

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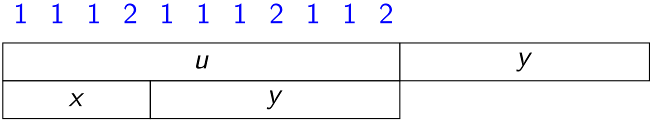

1 1 1 2 1 1 1 2 1 1 2

u y

x y

Génération de droites discrètes par substitutions et mots de Christoffel

Generation of a Sturmian word by morphisms

$$\alpha = [z_0; z_1, z_2, \ldots] = z_0 + \cfrac{1}{z_1 + \cfrac{1}{z_2 + \cfrac{1}{z_3 + \ldots}}}$$

Generation of a Sturmian word by morphisms

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$$\tau_n = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12^n \end{cases}$$

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For each $n \geq 0$, let $s_n = \tau_{z_0} \circ \tau_{z_1} \circ \cdots \circ \tau_{z_n}(2)$.

Sturmian word w_1 and w_2 of slope α is given by :

$$w_1 = \lim_{n \to \infty} s_{2n} \text{ and } w_2 = \lim_{n \to \infty} s_{2n+1}.$$

Euclid's algorithm

Computation of $[z_0; z_1, z_2, \ldots]$ from $\alpha = b/a$.

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- Let $u_0 = b$ and $u_1 = a$,
- For $i \geq 0$, (while $u_{i+1} > 0$)
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First steps :

$$u_2 = u_0 - \left\lfloor \frac{u_0}{u_1} \right\rfloor u_1,$$

$$u_3 = u_1 - \left\lfloor \frac{u_1}{u_2} \right\rfloor u_2,$$

$$\vdots$$

Euclid's algorithm, matrix form

Computation of $[z_0; z_1, z_2, \ldots]$ from $\alpha = b/a = b_0/a_0$.

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} -z_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \text{ where } z_n = \left\lfloor \frac{b_n}{a_n} \right\rfloor$$

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If $a_n \neq 0$ let $M_{z_n} = \begin{bmatrix} 0 & 1 \\ 1 & z_n \end{bmatrix}$, otherwise $M_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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$$\begin{bmatrix} a \\ b \end{bmatrix} = M_{z_1} M_{z_2} \cdots M_{z_n} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

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Euclid's algorithm, matrix form

Computation of $[z_0; z_1, z_2, \ldots]$ from $\alpha = b/a = b_0/a_0$.

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$$\begin{bmatrix} a \\ b \end{bmatrix} = \lim_{n \to \infty} M_{z_1} M_{z_2} \cdots M_{z_n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Summary

In order to draw a discrete line of slope $\alpha = b/a$:

- Compute the matrices $M_{z_n} = \begin{bmatrix} 0 & 1 \\ 1 & z_n \end{bmatrix}$ in order to obtain the list $[z_0; z_1, z_2, \ldots]$.
- Compute a Sturmian word w_α using the morphisms

$$\tau_{z_n} = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12^{z_n} \end{cases}$$

- Draw the geometric representation of w_α.

Formalization of the Freeman chain-code

Let $\mathcal{A}_d = \{1, 2, \ldots, d\}$ and $(e_1, e_2, \ldots, e_d)$ be the canonical base of $\mathbb{R}^d$. We consider $\mathfrak{F}$ be the vector space of mappings from $\mathbb{Z}^d \times \mathcal{A}_d$ to $\mathbb{R}$ that takes everywhere zero value except for a finite set.

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Let $(\vec{x}, e_i)$ be the element of $\mathfrak{F}$ that takes value 1 at $(\vec{x}, i)$ and 0 elsewhere.

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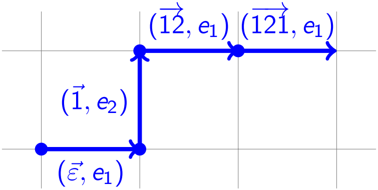

The E_1 operator

The 1-*dimensional geometric realization* $E_1(\sigma)$ of a word morphism σ is the linear mapping defined on $\mathfrak{F}$ such that :

$$\begin{tikzcd}
\mathcal{A}_d^* \arrow[r, "\sigma"] \arrow[d, "\pi_d"] & \mathcal{A}_d^* \arrow[d, "\pi_d"] \\
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$$E_1(\sigma)(\vec{x}, e_i) := \sum_{\substack{uj \text{ prefix of } \sigma(i) \\ u \in \mathcal{A}_d^*, j \in \mathcal{A}_d}} (M_\sigma \vec{x} + \vec{u}, e_j)\,.$$

The E_1^* operator

We consider $\mathfrak{F}^*$ the dual space of $\mathfrak{F}$ and the linear form :

$$\langle (\vec{y}, e_j), (\vec{x}, e_i^*) \rangle \overset{\text{def.}}{=} \begin{cases} 1 \text{ if } \vec{x} = \vec{y} \text{ and } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

The dual operator E_1^* of E_1 is given by

$$\langle E_1(\sigma)(\vec{y}, e_j)\,,\,(\vec{x}, e_i^*) \rangle = \langle (\vec{y}, e_j)\,,\,E_1^*(\sigma)(\vec{x}, e_i^*) \rangle\,.$$

In the case where M_σ is unimodular

$$E_1^*(\sigma)(\vec{x}, e_i^*) := \sum_{j \in \mathcal{A}} \sum_{ui \text{ prefix of } \sigma(j)} \left(M_\sigma^{-1}\left(\vec{x} - \vec{u}\right), e_j^*\right).$$

Geometrical representation of $\mathfrak{F}^*$

We represent an element $(\vec{x}, e_i^*)$ as :

$$(\vec{x}, e_i^*) \longrightarrow \{\vec{x} + e_i + \sum_{i \neq j} \lambda_j e_i \mid \lambda_j \in [0,1]\}.$$

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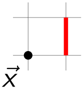
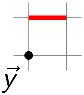
Examples :

- $d = 2$



$(\vec{x}, e_1^*)$



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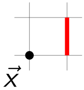

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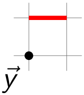
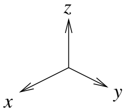
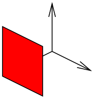


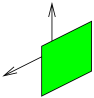
$(\vec{y}, e_2^*)$

- $d = 3$





$(\vec{0}, e_1^*)$



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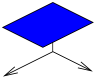

$(\vec{0}, e_3^*)$

Composition of E_1 and E_1^*

Definition

A substitution σ is *primitive* if, for any letter $i \in \mathcal{A}$ there exists n such that $\sigma^n(i)$ contains all the letters of $\mathcal{A}$.

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Given σ and τ two primitive substitutions :

$$E_1(\sigma) \circ E_1(\tau) = E_1(\sigma \circ \tau),$$

$$E_1^*(\sigma) \circ E_1^*(\tau) = E_1^*(\tau \circ \sigma).$$

Example in dimension 2

Given some slope $\alpha = b/a = [z_0; z_1, z_2, \dots]$ let's take a look at the geometric representations

- $E_1(\tau_{z_0}) \circ E_1(\tau_{z_1}) \circ \cdots \circ E_1(\tau_{z_n})(\vec{0}, e_2)$,

and

- $E_1^*(\tau_{z_0}) \circ E_1^*(\tau_{z_1}) \circ \cdots \circ E_1^*(\tau_{z_n})(\vec{0}, e_2)$,

for $n = 1, 2, \dots$

Recall that :

$$\begin{bmatrix} q_n \\ p_n \end{bmatrix} = M_{z_1} M_{z_2} \cdots M_{z_n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\frac{p_n}{q_n} = [z_0; z_1, z_2, \dots, z_n]$$

Example in dimension 2

$$(a, b) = (\pi, \sqrt{3}),\ \sqrt{3}/\pi = [0; 1, 1, 4, 2, 1, 2, 3, 7, 3, \dots],$$

$$E_1(\tau_0)(\vec{0}, e_2)$$

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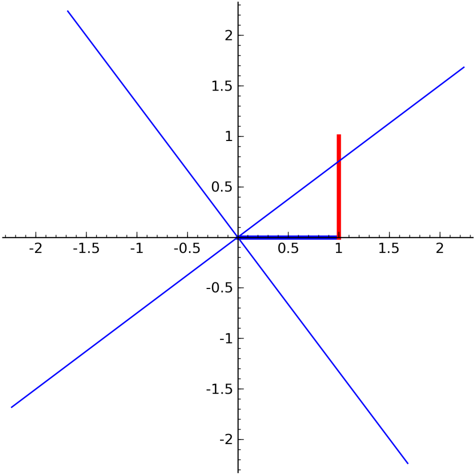
$$\tau_n = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12^n \end{cases}$$



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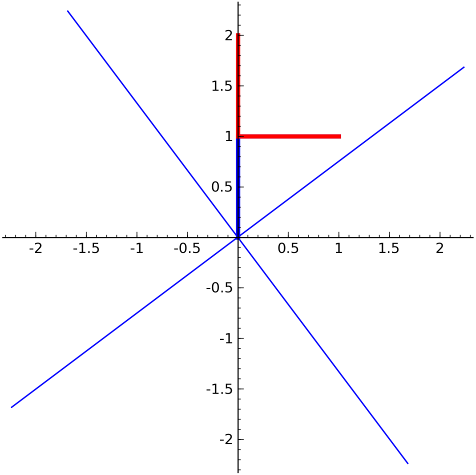

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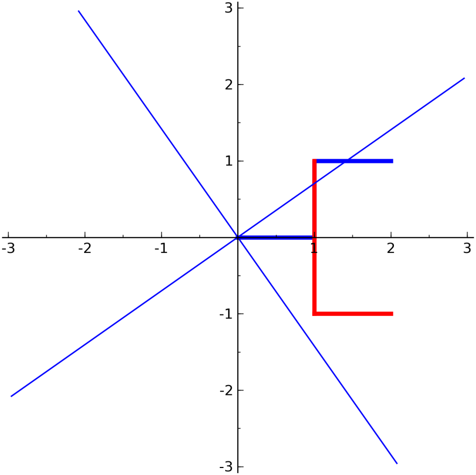

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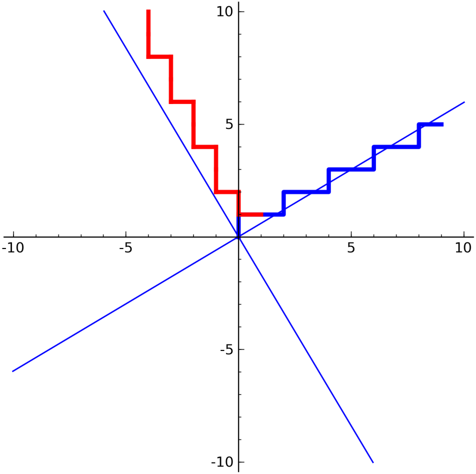

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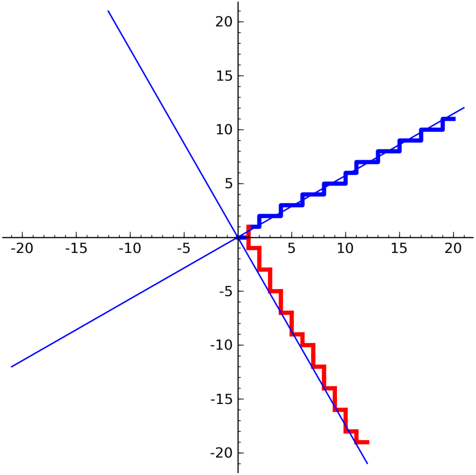

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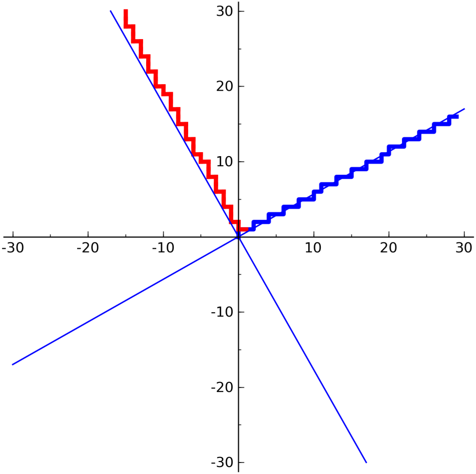

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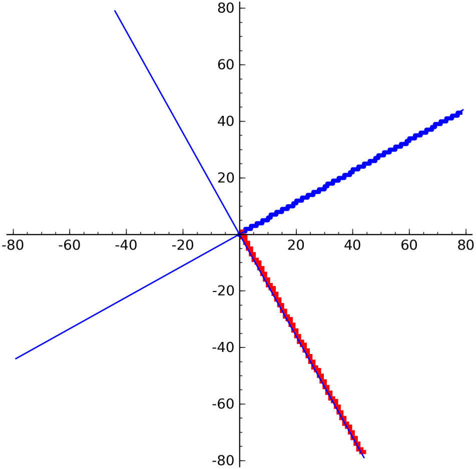

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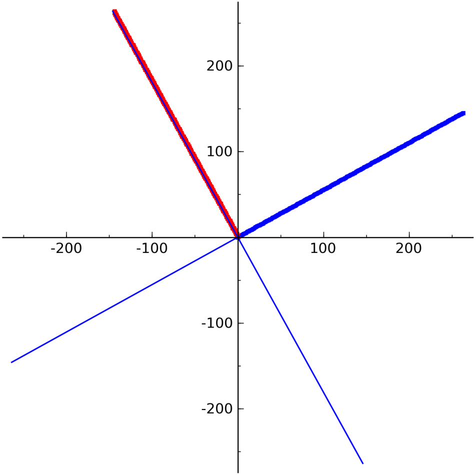

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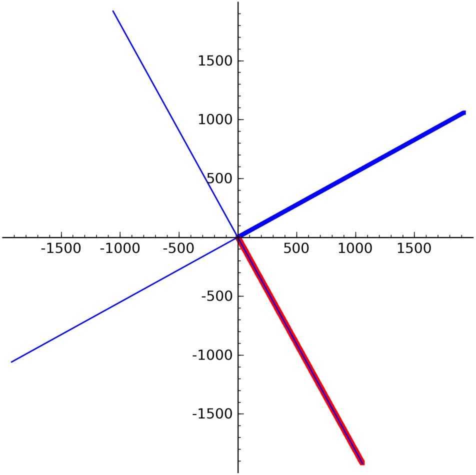

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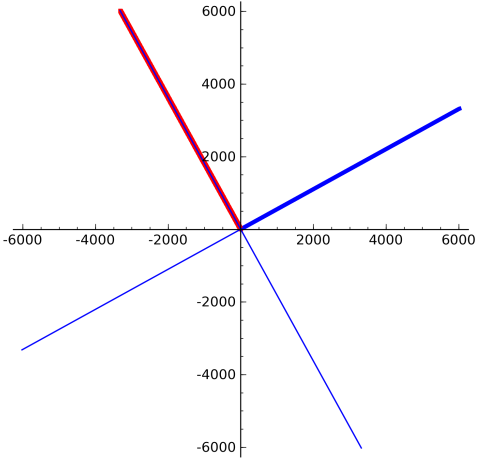

E_1^* and Christoffel words

Theorem (Berthé, de Luca, Reutenauer, 2007)

The geometrical representation of

$$E_1^*(\tau_{z_0}) \circ E_1^*(\tau_{z_1}) \circ \cdots \circ E_1^*(\tau_{z_n})(\vec{0}, e_2^*)$$

codes the Christoffel word of slope p_n/q_n *where* $q_n/p_n = [z_0; z_1, \ldots, z_n]$.

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Corollary

Given $\alpha = [z_0; z_1, z_2 \ldots] \notin \mathbb{Q}$, *limit*

$$\lim_{n\to\infty} E_1^*(\tau_{z_0}) \circ E_1^*(\tau_{z_1}) \circ \cdots E_1^*(\tau_{z_n})\big((\vec{0}, e_1^*) + (\vec{0}, e_2^*)\big)$$

exist and its geometrical interpretation is a discrete line of slope $-1/\alpha$.

The primitive and unimodular case

Definition

A substitution σ is *unimodular* if its incidence matrix M_σ has determinant $+1$ or -1.

Let $\mathcal{P}_{\vec{\alpha}}$ be the *plane* that passes through the origin with normal vector $\vec{\alpha}$.
Let $\mathfrak{G}_{\vec{\alpha}} = \{(\vec{x}, e_i^*) \in \mathfrak{F}^* \mid \mathcal{P}_{\vec{\alpha}}$ intersects the segment $[\vec{x}, \vec{x} + e_i]\}$

Theorem (Arnoux, Ito)

If σ is a primitive unimodular substitution, then

$$E_1^*(\sigma)(\mathfrak{G}_{\vec{\alpha}}) = \mathfrak{G}_{{}^tM_\sigma\vec{\alpha}}$$

Moreover, two distincs elements $(\vec{x}, e_i^*), (\vec{y}, e_j^*)$ have disjoint images by $E_1^*(\sigma)$.

Discrete plane generation

Given a vector $\vec{\alpha}$, in order to generate the discrete plane $\mathfrak{G}_{\vec{\alpha}}$:

- How to generate a sequence of primitive unimodular substitutions $(\sigma_i)_{i\geq 1}$ such that $\vec{\alpha} = \lim_{n\to\infty} {}^tM_{\sigma_1}{}^tM_{\sigma_2}\cdots{}^tM_{\sigma_n}e_3$?

Continued fraction

Many possible generalizations : Jacobi-Perron, Brun, Poincaré, Selmer, ...

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- *Best approximation* Let $\alpha = [z_0; z_1, \dots]$. For each $n \geq 0$, let $\frac{p_n}{q_n} = [z_0; z_1, z_2, \dots, z_n]$ then any $p, q \in \mathbb{N}$ such that $1 \leq q \leq q_n$ and $\frac{p}{q} \neq \frac{p_n}{q_n}$ satisfies

$$|q_n\alpha - p_n| < |q\alpha - p| .$$

See, e.g., Khintchine, Cassels.

Continued fraction

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- Let $\alpha, \beta \in \mathbb{R}$ the Jacobi-Perron algorithm computes $(p_i, q_i, r_i)_{i\geq 1}$ such that for all $n \geq 0$,

$$\left|\alpha - \frac{p_n}{r_n}\right| < \frac{1}{r_n^{1+\epsilon}} \text{ and } \left|\beta - \frac{q_n}{r_n}\right| < \frac{1}{r_n^{1+\epsilon}}.$$

Jacobi-Perron's algorithm

- Input : $(a, b, c) \in \mathbb{R}^3$, $0 \le \min(a, b), \max(a, b) \le c$
  Initialization : $(a_0, b_0, c_0) := (a, b, c)$,

$$(a_{n+1}, b_{n+1}, c_{n+1}) := \begin{cases} \left(b_n - a_n \left\lfloor \frac{b_n}{a_n} \right\rfloor, c_n - a_n \left\lfloor \frac{c_n}{a_n} \right\rfloor, a_n\right) \text{ if } a_n \neq 0, \\ \\ \left(0, c_n - b_n \left\lfloor \frac{c_n}{b_n} \right\rfloor, b_n\right) \text{ if } a_n = 0 \text{ and } b_n \neq 0, \\ \\ (0, 0, c_n) \text{ if } a_n = b_n = 0. \end{cases}$$

Jacobi-Perron's algorithm

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The *Jacobi-Perron matrices* are the unimodular matrices that satisfy :

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = {}^t M_n \begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{bmatrix}$$

Jacobi-Perron matrices

- $(a_{n+1}, b_{n+1}, c_{n+1}) := (b_n - a_n B_n, c_n - a_n C_n, a_n)$

  $$M_{B_n,C_n} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B_n & C_n \end{bmatrix} \text{ with } B_n = \left\lfloor \frac{b_n}{a_n} \right\rfloor \text{ and } C_n = \left\lfloor \frac{c_n}{a_n} \right\rfloor,$$

- $(a_{n+1}, b_{n+1}, c_{n+1}) := (0, c_n - b_n E_n, b_n)$,

  $$M_{E_n} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & E_n \end{bmatrix} \text{ with } E_n = \left\lfloor \frac{c_n}{b_n} \right\rfloor,$$

- $(a_{n+1}, b_{n+1}, c_{n+1}) := (0, 0, c_n)$, $M_{\mathrm{Id}} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Jacobi-Perron substitutions

- $M_{B_n,C_n} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B_n & C_n \end{bmatrix}$

- $M_{E_n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & E_n \end{bmatrix}$

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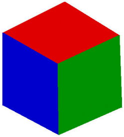
Notation : we note $(M_{<i>})_{i\geq 1}$ (resp. $(\sigma_{<i>})_{i\geq 1}$) the sequence of matrices (resp. substitutions) produced by the Jacobi-Perron algorithm.

Examples in dimension 3

Starting with the unit cube $\mathcal{U} = (\vec{0}, e_1^*) + (\vec{0}, e_2^*) + (\vec{0}, e_3^*)$.

$$\mathcal{U}$$

$$\sigma_{B,C} = \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 13^{B_n} \\ 3 \mapsto 23^{C_n} \end{cases}$$



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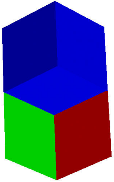

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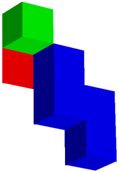

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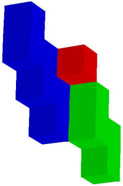

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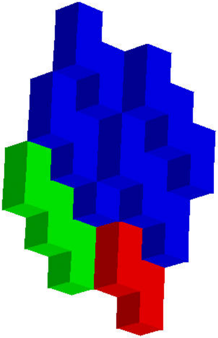

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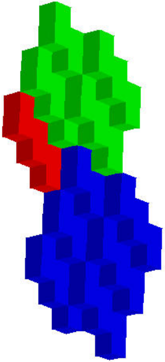

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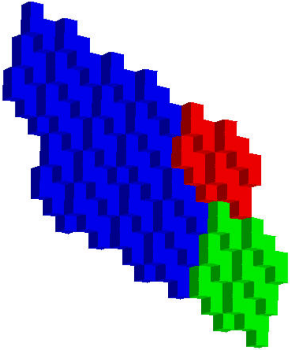

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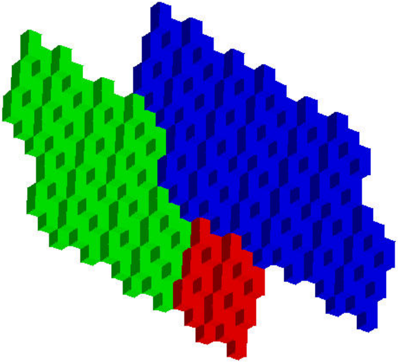

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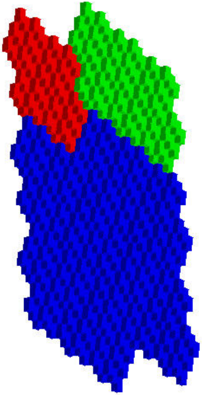

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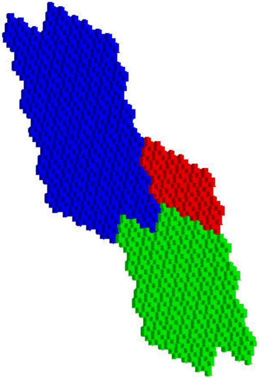

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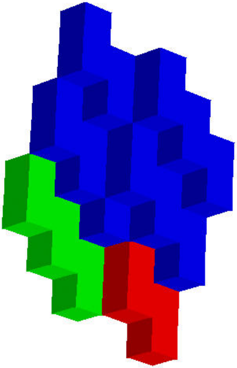

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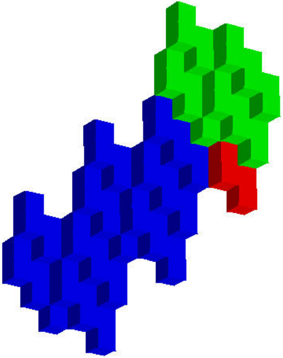

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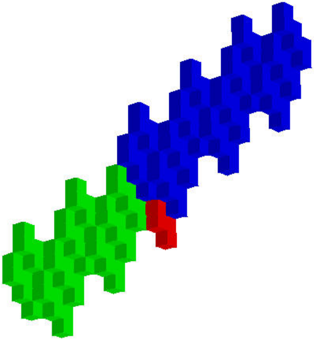

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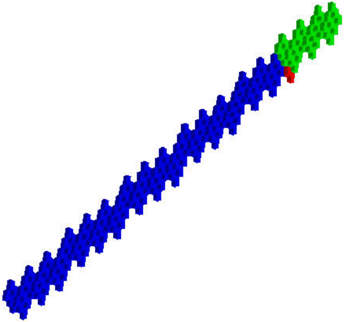

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$$E_1^*(\sigma_{1,1})(\mathcal{U})$$

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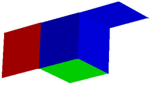

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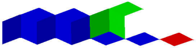

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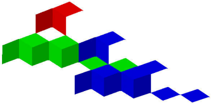

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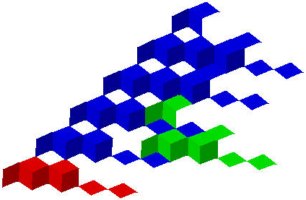

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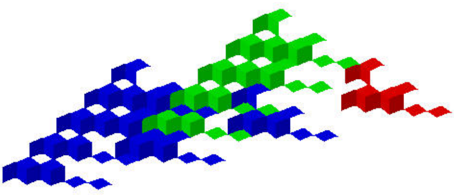

Generating the whole plane $\mathfrak{G}_{\vec{\alpha}}$

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- $\dim_{\mathbb{Q}}(a, b, c) = 1$
  There exists N such that for all $n \geq N$, $\sigma_{<n>} = \sigma_{\mathrm{Id}}$.
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- $\dim_{\mathbb{Q}}(a, b, c) = 2$
  There exists N such that for all $n \geq N$, $\sigma_{<n>} = \sigma_{E_n}$.
  $\lim\limits_{n\to\infty} E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<2n>})((\vec{0}, e_2^*) + (\vec{0}, e_3^*))$ is an infinite stripe.

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- $\dim_{\mathbb{Q}}(a, b, c) = 3$
  For all $n \geq 1$, $\sigma_{<n>} = \sigma_{B_n, C_n}$.
  As n grows, $E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<2n>})(\vec{0}, e_3^*)$ forms some infinite *potato*.

Generating the whole plane $\mathfrak{G}_{\vec{\alpha}}$, with $\dim_{\mathbb{Q}}(a, b, c) = 3$

- ⋆ There exists $n_0 \geq 1$ such that for all $k \geq 0$
  - $B_{n_0+3k} = C_{n_0+3k}$,
  - $C_{n_0+3k+1} - B_{n_0+3k+1} \geq 1$,
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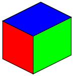
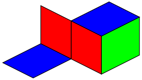
$\mathcal{U} :=$



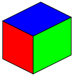
$\mathcal{V} :=$

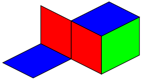


Generating the whole plane $\mathfrak{G}_{\vec{\alpha}}$, with $\dim_{\mathbb{Q}}(a, b, c) = 3$

⋆ There exists $n_0 \geq 1$ such that for all $k \geq 0$

- $B_{n_0+3k} = C_{n_0+3k}$,
- $C_{n_0+3k+1} - B_{n_0+3k+1} \geq 1$,
- $B_{n_0+3k+2} = 0$.

$\mathcal{U} :=$ 

$\mathcal{V} :=$ 

Theorem (Ito, Ohtsuki)

Given $\dim_{\mathbb{Q}}(a, b, c) = 3$,
If the condition $\star$ *holds then*

$$\mathfrak{G}_{(a,b,c)} = \lim_{n\to\infty} E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{V}).$$

Otherwise,

$$\mathfrak{G}_{(a,b,c)} = \lim_{n\to\infty} E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{U}).$$

Connectedness

$$\mathcal{W} = \begin{cases} (\vec{0}, e_3^*) \text{ if } \dim_{\mathbb{Q}}(a,b,c) \leq 2, \\ \mathcal{U} \text{ if } \dim_{\mathbb{Q}}(a,b,c) = 3 \text{ and not } \star, \\ \mathcal{V} \text{ if } \dim_{\mathbb{Q}}(a,b,c) = 3 \text{ and } \star\,. \end{cases}$$

Theorem (Berthé, Lacasse, Paquin, P.)

For any $n \geq 1$, the pattern $\mathcal{T}_n = E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{W})$ is a simply connected set.

Polyamond patterns

Let π_0 be the orthogonal projection on the plane $\mathcal{P}_0 : x + y + z = 0$.

Definition

A pattern $\mathcal{X}$ is a *polyamond pattern* if the topological boundary of its projection $\pi_0(\mathcal{X})$ is a Jordan curve.

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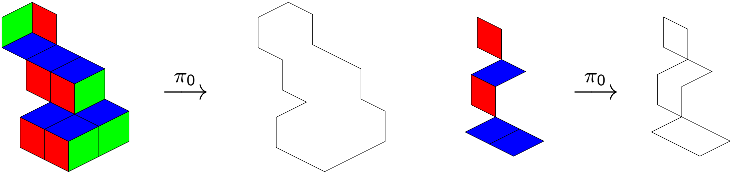

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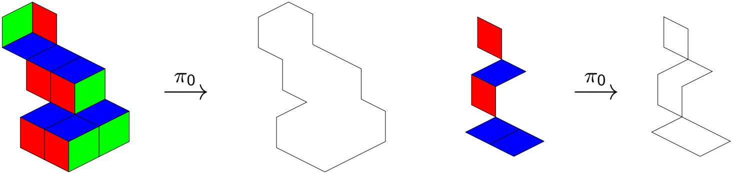
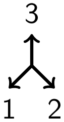


Proposition

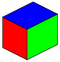
A polyamond pattern $\mathcal{X}$ is simply connected.

Eight-curves

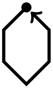
$$\overline{\mathcal{A}}_3 = \{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$$



$\mathcal{U}$:



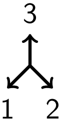
$\xrightarrow{\pi_0}$

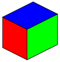
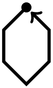


$1\bar{3}2\bar{1}3\bar{2}$ ← boundary word

Eight-curves

$$\overline{\mathcal{A}}_3 = \{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$$



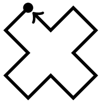
$\mathcal{U}$:  $\xrightarrow{\pi_0}$  $1\bar{3}2\bar{1}3\bar{2} \leftarrow$ boundary word

Definition

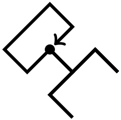
Given a word $w \in \overline{\mathcal{A}}_d^*$

- w is a *closed curve* if $\vec{w} = \vec{0}$ and $\pi_{F_d}(w) \neq \varepsilon_{F_d}$, where π_{F_d} is the canonical projection from $\overline{\mathcal{A}}_d^*$ to the free group F_d.
- w is an *eight-curve* if it is a closed curve and if it admits a conjugate of the form $w \equiv uv$ where u and v are closed curves.

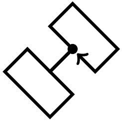
Examples



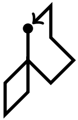
$$\underbrace{1212\bar{1}2\bar{1}\bar{2}\bar{1}\bar{2}1\bar{2}}_{\vec{0}}$$



$$\underbrace{2\bar{1}2\bar{2}112\bar{2}\bar{1}\bar{2}}_{\vec{0}}\,\underbrace{1\bar{2}\bar{1}\bar{1}21}_{\vec{0}}$$



$$\underbrace{121\bar{2}\bar{2}\bar{1}2\bar{1}}_{\vec{0}}\,\underbrace{\bar{2}\bar{1}2212\bar{2}}_{\vec{0}}$$



$$\bar{3}\,\underbrace{1\bar{3}\bar{1}3}_{\vec{0}}\,2\bar{1}\bar{2}31$$

Eight-curves and connectedness

Proposition

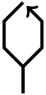
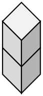
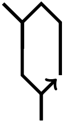
The boundary word of a pattern $\mathcal{X}$ that is not a polyamond pattern is an eight-curve.

Theorem (Ei)

Let $\mathcal{X}$ be a pattern with boundary word w and σ be a primitive unimodular substitution, then $\widetilde{\sigma^{-1}}(w)$ is a boundary word for $E_1^*(\sigma)(\mathcal{X})$.

Example

Let σ be the Tribonacci substitution : $\sigma(1) = 12$, $\sigma(2) = 13$,$\sigma(3) = 1$,

k	$E_1^*(\sigma^k)(\vec{0}, e_1^*)$	$\left(\widetilde{\sigma^{-1}}\right)^k (23\bar{2}\bar{3})$
0		 $23\bar{2}\bar{3}$
1		 $1\bar{3}2\bar{3}3\bar{1}3\bar{2}$
2		 $33\bar{2}1\bar{3}3\bar{2}2\bar{3}\bar{3}2\bar{3}3\bar{1}$

Idea of the proof

We define some set of forbidden words E such that given any substitution σ obtained by the Jacobi-Perron's algorithm

- Eight-curves are included in E.
- The boundary words of the generating patterns are not in E.
- For any word w, $\widetilde{\sigma^{-1}}(w) \in E$ implies that $w \in E$.

Conclusion

	Mots de Christoffel	Patates de Jacobi-Perron
Approximations successives	✓	✓
Génère tout	✓	✓
Simple connexité	n/a	✓
Récurrence	✓	✓
Périodes	✓	?
Palindrômes	✓	?
Interprétation géométrique (convexité)	✓	?

MERCI !