# Génération de plans discrets par des algorithmes de fractions continues, le cas rationnel

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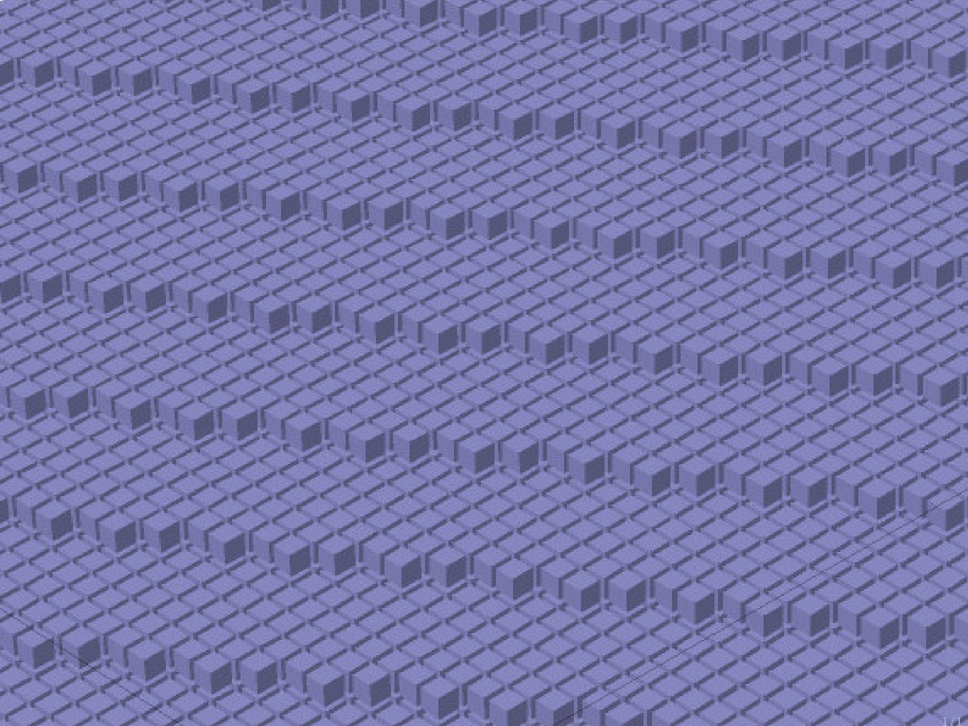
Dyna3S

Jeudi le 10 novembre, Paris



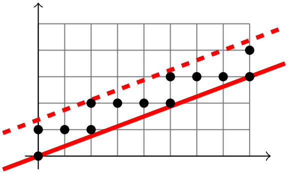
**Definition (Reveillès (1991), Kovalev (1990))**

An **arithmetic digital line** is the set :

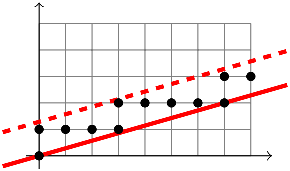
$$\mathcal{D}((a,b),\mu) = \{(x,y) \in \mathbb{Z}^2 \mid 0 \leq ax + by + \mu < |a| + |b|\}$$

where

- $(a, b)$ is the **normal vector**,
- $-b/a$ is the **slope**,
- $\mu$ is the **shift**.



$\mathcal{D}((-3,8),0)$

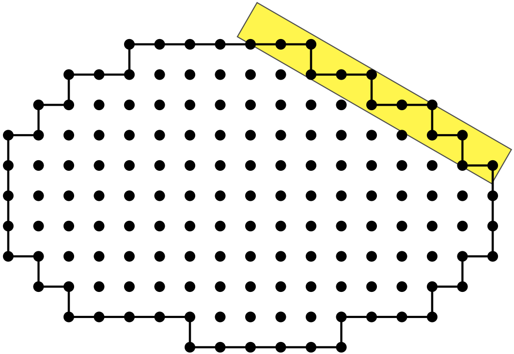


$\mathcal{D}((-2,7),0)$

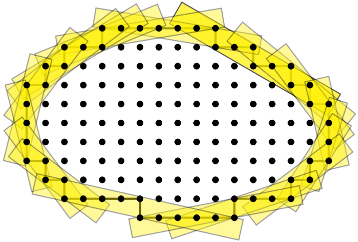
# Digital Straight Segment (DSS)

**Definition**

A **digital straight segment** is a finite and connected subset of a digital line.



# Tangential cover



**Definition** ([Feschet, Tougne 99])

The tangential cover of a discrete shape is the sequence of all maximal DSS on its boundary.

**Theorem** ([Debled-Rennesson, Reveilles 1995][Lachaud, vialard, de Vieilleville 2007])

*The computation of the tangential cover take a time in $\mathcal{O}(n)$ where $n$ is the number of points on the boundary of the shape.*

Applications of the tangential cover include :

- Convexity test [Debled-Rennesson, Reiter-Doerksen 04]
- Tangent estimation [Feschet, Tougne 99], [Lachaud, de Vieilleville 07]
- Length estimation [Lachaud, de Vieilleville 07]
- Curvature estimation [Lachaud, Kerautret, Naegel 08]
- Automatic noise detection [Lachaud, Kerautret 12]

**Theorem ([Lachaud, Kerautret 2012])**

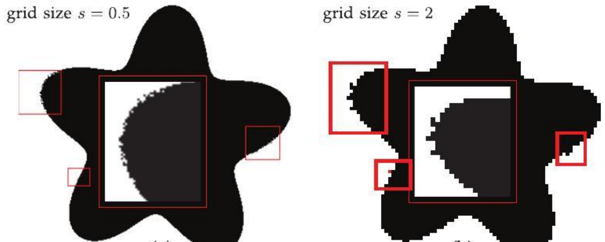
*Let $S$ be a simply conected shape in $\mathbb{R}^2$ with a piecewise $C^3$ boundary. Let $(L_j^h)$ be the lengths of DSS covering a point $P$ on $Dig_h(S)$, then :*

- *If $P$ is in a strictly convex or concave area:*

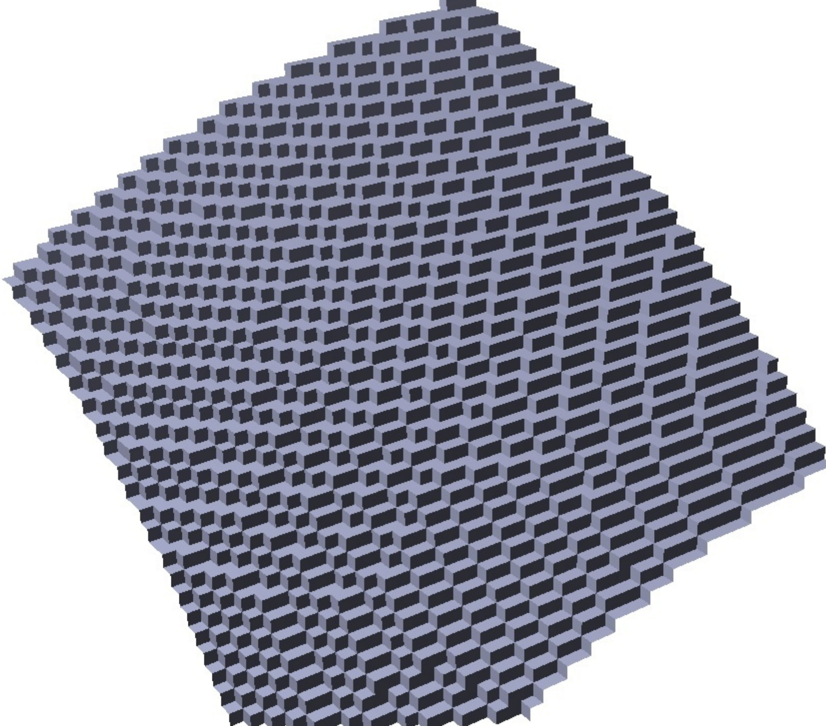
$$\Omega(1/h^{1/3}) \leq L_j^h \leq \mathcal{O}(1/h^{1/2}).$$

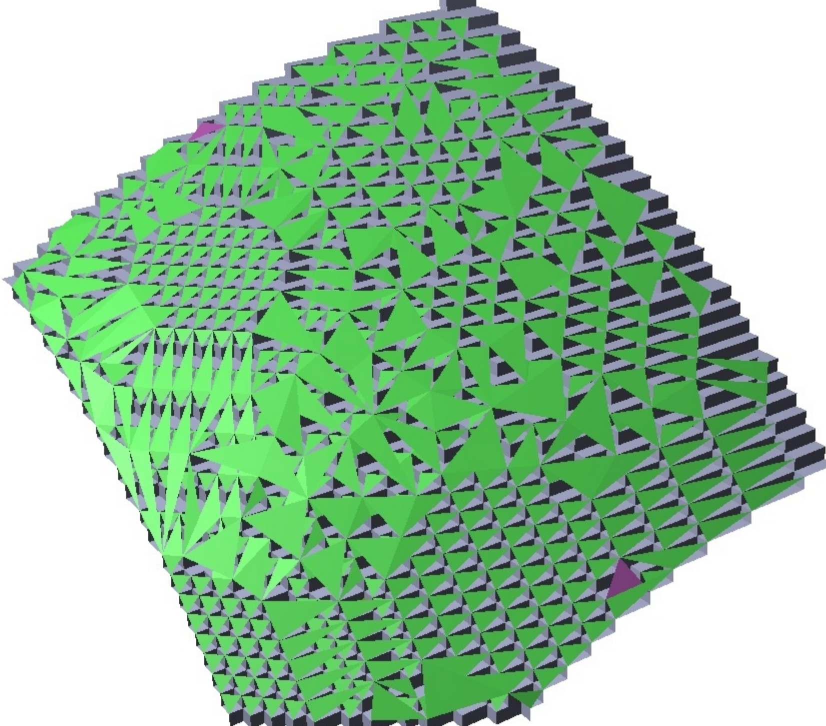
- *If $P$ is in a null curvature area:*

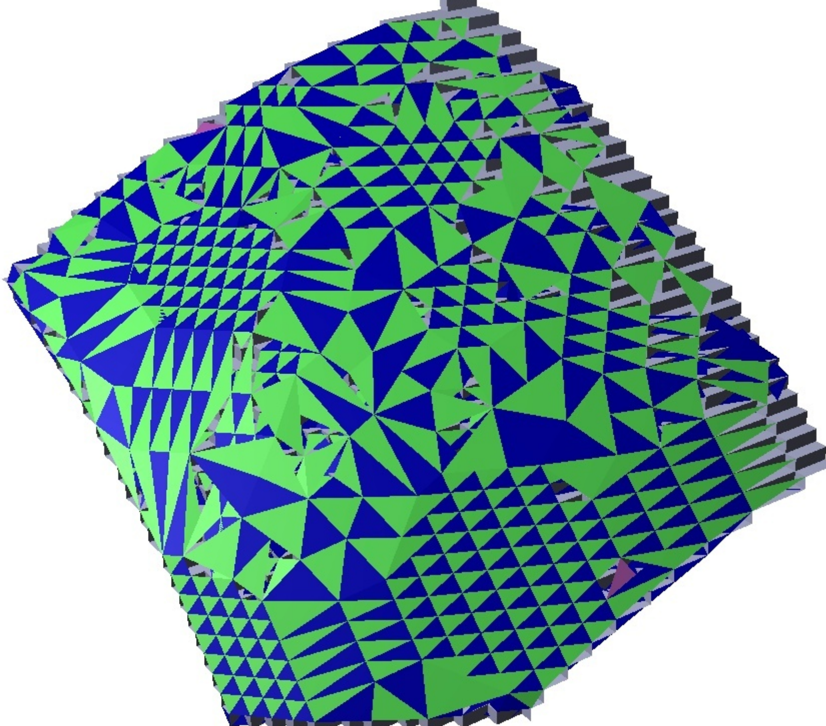
$$\Omega(1/h) \leq L_j^h \leq \mathcal{O}(1/h).$$



(a) (b)







# Digital lines and planes

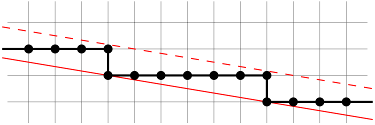
### Definition ([Reveillès 91])

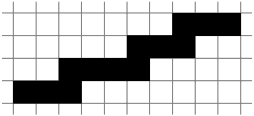
The **digital line/plane/hyperplane** $\mathcal{P}(v, \mu, \omega)$ with **normal vector** $v \in \mathbb{Z}^d$, **thickness** $\omega \in \mathbb{N}$ and **shift** $\mu \in \mathbb{R}$ is the subset of $\mathbb{Z}^d$ defined by:

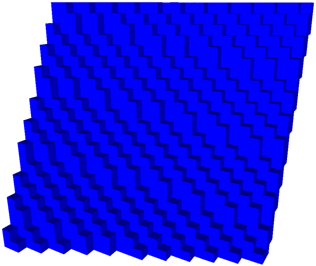
$$\mathcal{P}(v, \mu, \omega) = \left\{ x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle - \mu < \omega \right\}$$

$\mathcal{P}((1,6),7,0)$

$0 \leq 1x + 6y < 7$







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$\mathcal{P}(\ (1,6),\ 7\ )$

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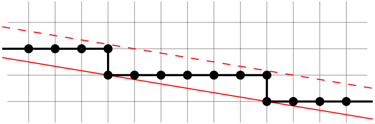
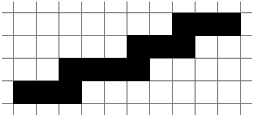
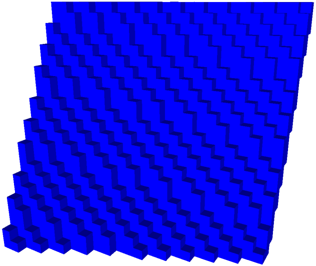


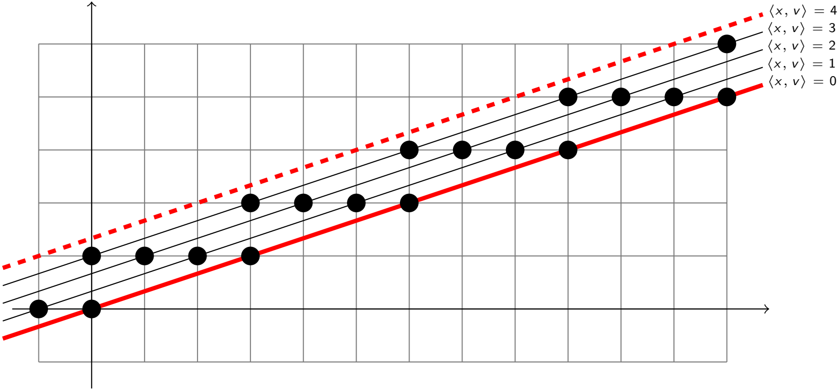




## Periodic structure of a digital line

Example with $v = (-3, 1)$:

- $\langle x, v \rangle$ is the **height** of $x$,
- $\mathcal{P}(v, 4) = \{x \in \mathbb{Z}^2 \mid 0 \leq \langle x, v \rangle < 4\}$.



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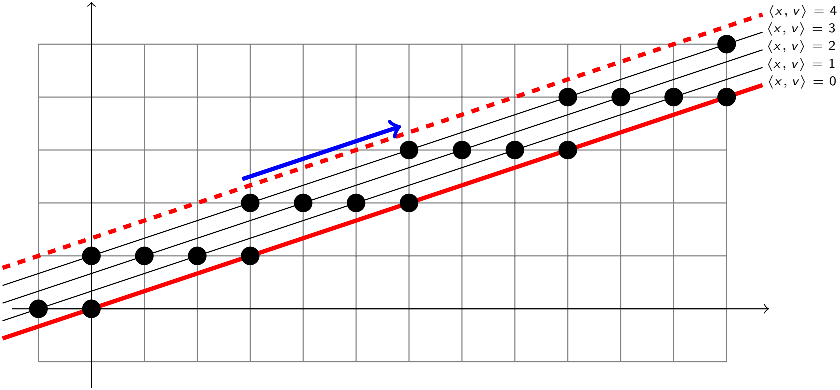
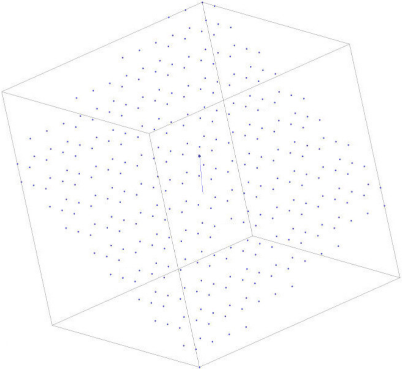
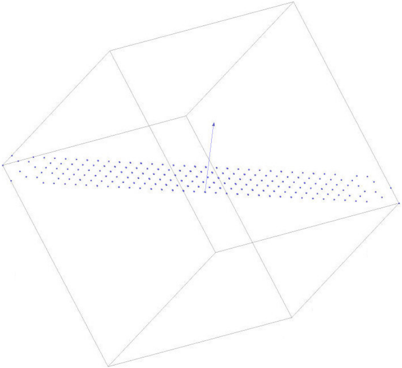



- $\langle x, v \rangle = \langle y, v \rangle \implies y - x$ is a period vector.
- A point of each height from 0 to $\|v\|_1 - 1$ appear in a period.

# Periodic structure of a digital plane

$v = (1, 2, 3), \quad \mathcal{P}(v, 6) = \{x \in \mathbb{Z}^3 \mid 0 \leq \langle x, v \rangle < 6\}$





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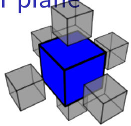

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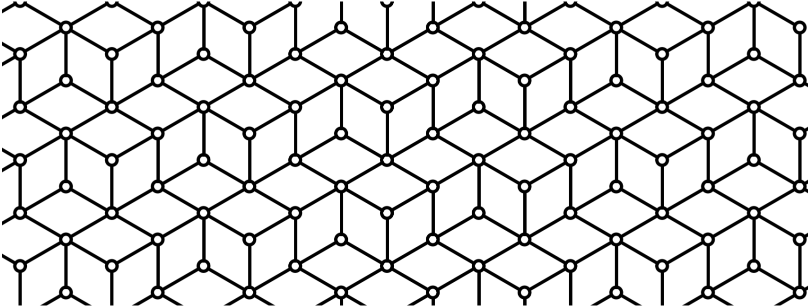

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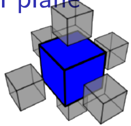

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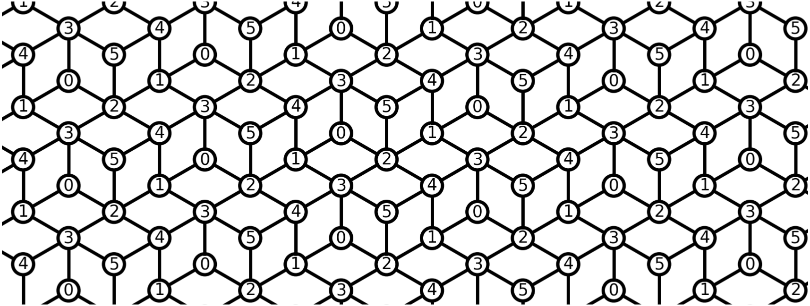

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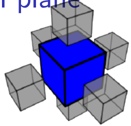

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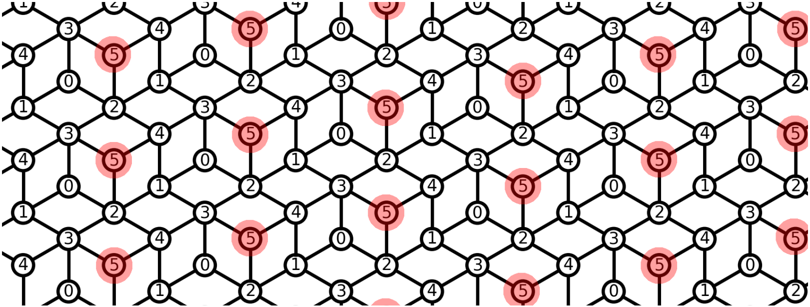

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- A point of each height from 0 to $\|v\|_1 - 1$ appears in a period.
- $\langle x, v \rangle = \langle y, v \rangle = \langle z, v \rangle \implies (y - x) \times (z - x) = \lambda v.$

## Periodic structure of a digital line

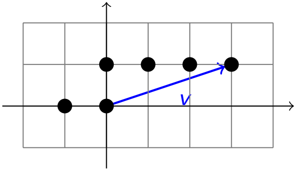
**Definition**

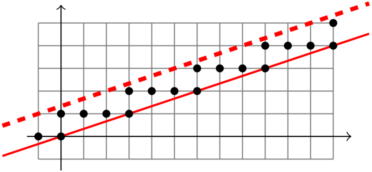
A set of points $S \subset \mathbb{Z}^d$ provided with a set of vectors $(b_i)_{i=1}^n \in \mathbb{Z}^d$ **spans** an infinite set $\Omega \subset \mathbb{Z}^d$ if

$$\Omega = \bigcup_{x \in \mathbb{Z}b_1 + \mathbb{Z}b_2 + \ldots + \mathbb{Z}b_n} (S + x).$$

(Like a tiling without a disjoint union.)

Example :

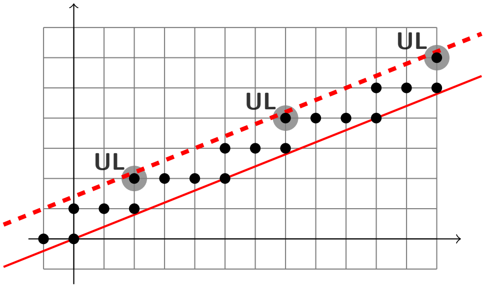
The set : 

provided with vector $v = (3, 1)$ spans the digital line $\mathcal{P}((-3, 1), 4)$.

# Main pattern of a digital line

- A point $x \in \mathcal{P}(v, \|v\|_1)$ is a **upper leaning point**, noted **UL**, if its height $\langle x, v \rangle$ is maximal.



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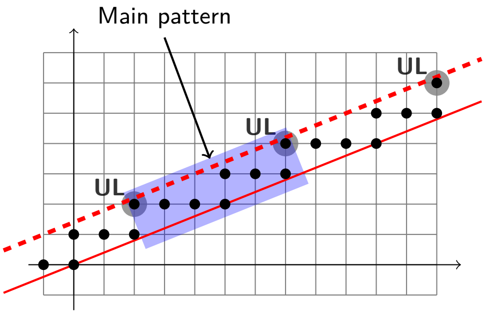

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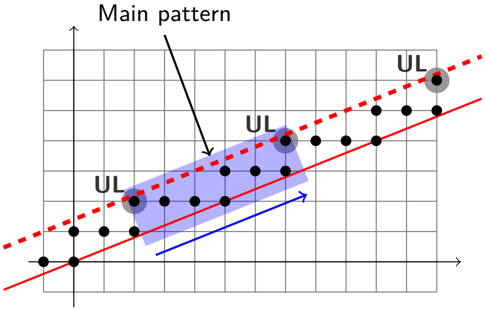
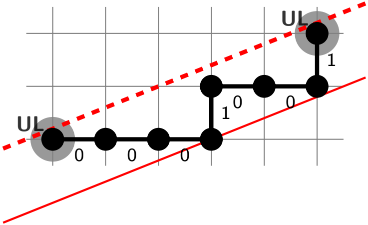
- Let $v$ be the vector defined by two consecutive **UL**, a main pattern provided with $v$ spans its digital line.



# Christoffel words

**Definition ([Christoffel 1875])**

A **Christoffel work** codes the digital immediately under the segment joining two integer points.




Christoffel word of **slope** $2/5$ : 0001001

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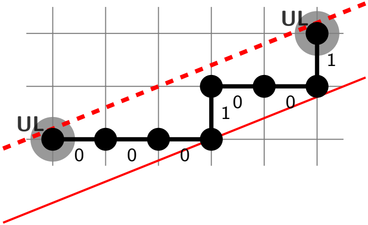


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### Theorem ([Borel, Laubie 93])

*Every Christoffel word, other than* $0$ *and* $1$*, is written in a* ***unique*** *way as a product of* ***two*** *Christoffel words.*

This is called the **standard factorization**, noted $w = (u, v)$.

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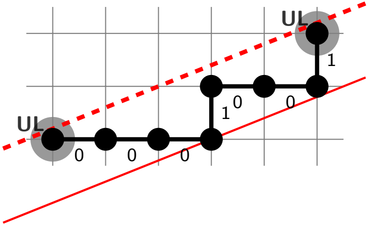


Christoffel word of **slope** $2/5$ : $0001001 = (0001, 001)$.

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## Christoffel Tree

If $(u, v)$ is a standard factorization, then $(u, uv)$ and $(uv, v)$ are standard factorizations of Christoffel words.

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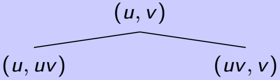
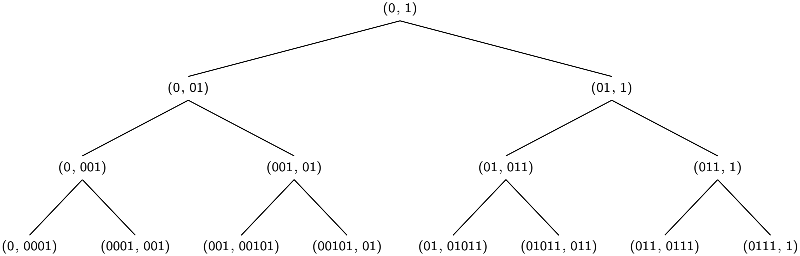
The **Christoffel Tree** is the tree obtained, starting from $(0, 1)$, using the rule :



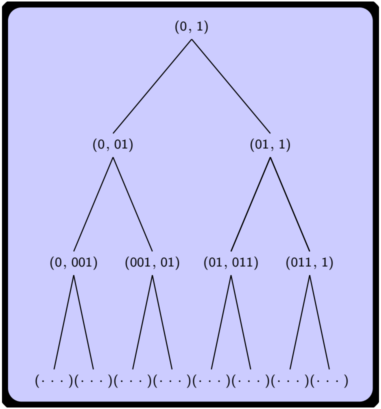

### Theorem

*Every Christoffel word appears exactly once in the Christoffel Tree.*

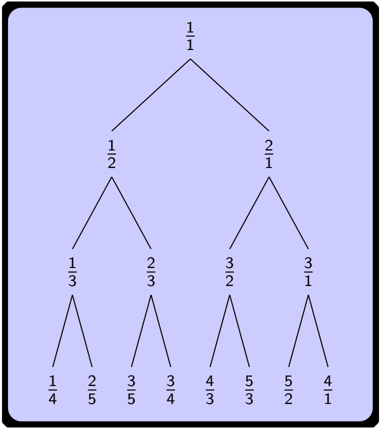


# Stern-Brocot Tree

Christoffel tree




Stern-Brocot tree.

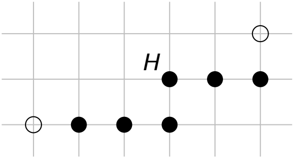



Every irreducible fraction appears exactly once in the Stern-Brocot tree.

# Main pattern of a digital line

- ○ : upper leaning points.
- Let $H$ be the highest point among {●}.

$(u, v)$ : slope $2/5$.



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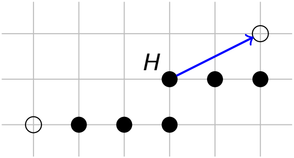

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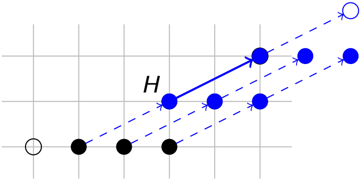

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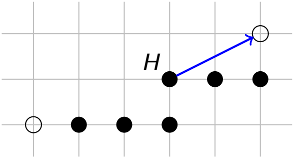


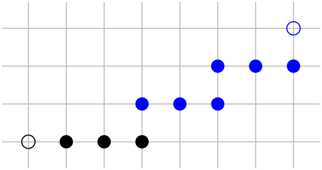

$(uv, v)$ : slope $3/8$.

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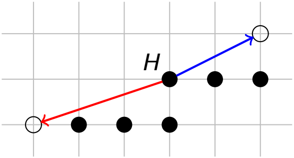
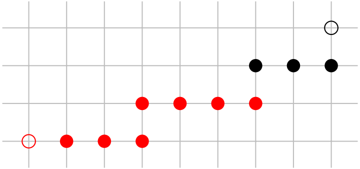
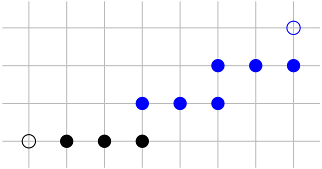


$(u, uv)$ : slope $3/7$.



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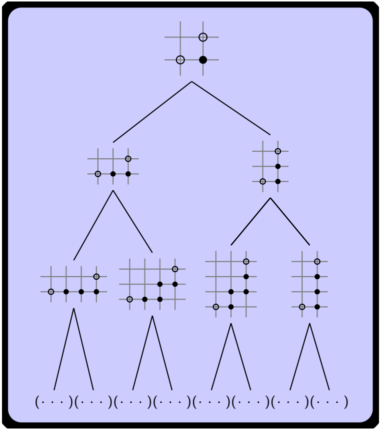
# Stern-Brocot Tree

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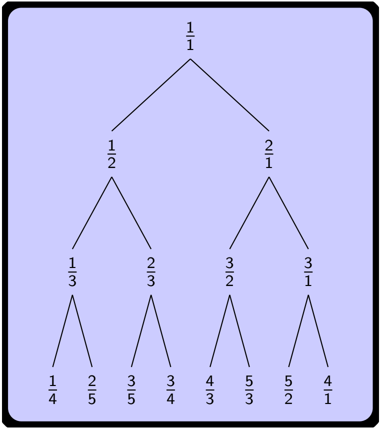
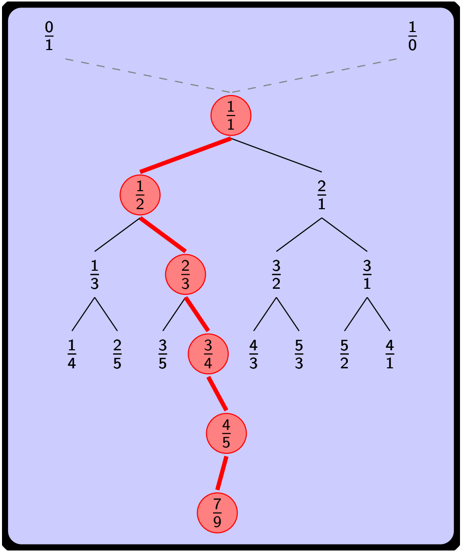



Every irreducible fraction appears exactly once in the Stern-Brocot tree.

# Euclid Algorithm

Stern-Brocot tree




| Euclid algorithm | Approximation |
|---|---|
| $(7, 9)$ | $(1, 1)$ |
| ↓ | ↓ |
| $(7, 2)$ | $(1, 2)$ |
| ↓ | ↓ |
| $(5, 2)$ | $(2, 3)$ |
| ↓ | ↓ |
| $(3, 2)$ | $(3, 4)$ |
| ↓ | ↓ |
| $(1, 2)$ | $(4, 5)$ |
| ↓ | ↓ |
| $(1, 1)$ | $(7, 9)$ |

# Matricial view

| | Euclid algorithm | Approx. |
|---|---|---|
| $n$ | $v_n$ | $a_n$ |
| 0 | $(\underline{7}, 9)$ | $(1, 1)$ |
| | $\downarrow$ | $\downarrow$ |
| 1 | $(7, \underline{2})$ | $(1, 2)$ |
| | $\downarrow$ | $\downarrow$ |
| 2 | $(5, \underline{2})$ | $(2, 3)$ |
| | $\downarrow$ | $\downarrow$ |
| 3 | $(3, \underline{2})$ | $(3, 4)$ |
| | $\downarrow$ | $\downarrow$ |
| 4 | $(\underline{1}, 2)$ | $(4, 5)$ |
| | $\downarrow$ | $\downarrow$ |
| 5 | $(1, 1)$ | $(7, 9)$ |

**Euclid algorithm**
Given a vector $(x, y)$, return

- $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ if $x < y$,
- $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ if $x > y$,
- **stop** if $x = y$.

Given a vector $v \in (\mathbb{N} \setminus \{0\})^2$, let :

- $v_0 = v$,
- For all $n \geq 1$ : $\begin{cases} M_n = \mathbf{Euclid}(v_{n-1}) \\ v_n = M_n v_{n-1}. \end{cases}$

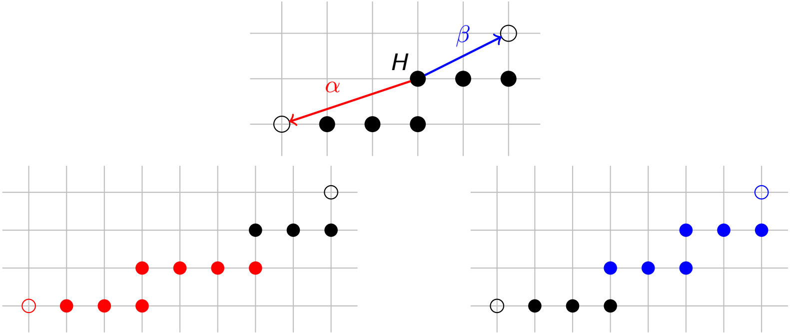
## Property

- $v_n = M_n M_{n-1} \cdots M_1 v$
- $a_n = M_1^{-1} M_2^{-1} \cdots M_n^{-1} \binom{1}{1}$

Let $UL_0$ and $UL_1$ be two upper leaning points of a main pattern of $\mathcal{P}(a_n, \|a_n\|_1)$ and $H$ be the Bezout point. Let $\alpha = UL_0 - H$ and $\beta = UL_1 - H$, then

$$M_1^\top M_2^\top \cdots M_n^\top = \left[\begin{array}{cc} \alpha & \beta \end{array}\right]$$

$$M_1^\top \cdots M_n^\top e_1 = \alpha, \qquad M_1^\top \cdots M_n^\top e_2 = \beta.$$



# The Translation-Union Construction

**Construction**
[Domenjoud,Vuillon 12],
[Berthé, Jamet, Jolivet, P. 2013]

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

$M_n$ : the matrix selected from $v_{n-1}$,

$v_n = M_n v_{n-1}$

$\delta_n$ : the index of the coordinate of $v_{n-1}$ that is subtracted,

$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}$, *(translation)*

$B_n = B_{n-1} \cup (T_n + B_{n-1})$, *(body)*

$H_n = \sum_{i \in \{1,\dots,n\}} T_i$, *(highest point)*

$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}$. *(legs)*

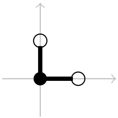
Note that:

$H_n \in B_n$,

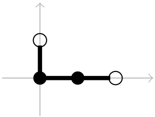
$L_n \cap B_n = \emptyset$.

$\bullet \in B_n$, $\circ \in L_n$

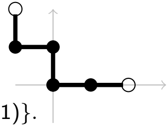
$v_0 = (2, 3)$,
$a_0 = (1, 1)$
$H_0 = (0, 0)$,
$L_0 = \{(1, 0), (0, 1)\}$.



$v_1 = (2, 1), \delta_1 = 1$
$a_1 = (1, 2)$
$T_1 = (1, 0)$
$H_1 = (1, 0)$,
$L_1 = \{(2, 0), (0, 1)\}$.



$v_2 = (1, 1), \delta_2 = 2$
$a_2 = (2, 3)$
$T_2 = (-1, 1)$
$H_2 = (0, 1)$,
$L_2 = \{(2, -1), (-1, 1)\}$.



## 3D continued fraction algorithms

**Euclid** algorithm : given two numbers subtract the smallest to the largest.

$(7,9) \to (7,2) \to (5,2) \to (3,2) \to (1,2) \to (1,1) \to (1,0)\circlearrowleft$

Given three numbers :

- **Selmer** : subtract the smallest to the largest.
  $(3,7,5) \to (3,4,5) \to (3,4,2) \to (3,2,2) \to (1,2,2) \to (1,2,0)\circlearrowleft$ .
- **Brun** : subtract the second largest to the largest.
  $(3,7,5) \to (3,2,5) \to (3,2,2) \to (1,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)\circlearrowleft$ .
- **Fully subtractive** : subtract the smallest to the two others.
  $(3,7,5) \to (3,4,2) \to (1,2,2) \to (1,1,1) \to (1,0,0)\circlearrowleft$ .
- **Poincaré** : subtract the smallest to the mid and the mid to the largest.
  $(3,7,5) \to (3,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)\circlearrowleft$ .
- **Arnoux-Rauzy** : subtract the sum of the two smallest to the largest (not always possible).
  $(3,7,5) \to \text{impossible.}$
- ...

# Example : Fully Subtractive $v = (6, 8, 11)$

**Construction**

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

$M_n$ : the matrix selected from $v_{n-1}$,

$v_n = M_n v_{n-1}$

$\delta_n$ : the index of the coordinate of $v_{n-1}$ that is subtracted,

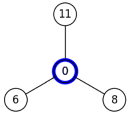
$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}$, *(translation)*

$B_n = B_{n-1} \cup (T_n + B_{n-1})$, *(body)*

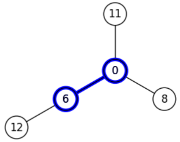
$H_n = \sum_{i \in \{1,\ldots,n\}} T_i$, *(highest point)*

$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}$. *(legs)*

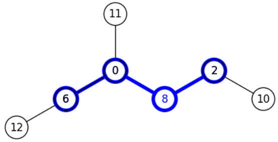
- Step 0 : $v_0 = (6, 8, 11)$, $a_0 = (1, 1, 1)$,




- Step 1 : $v_1 = (6, 2, 5)$, $a_1 = (1, 2, 2)$,

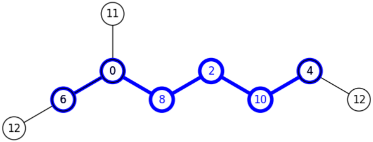



- Step 2 : $v_2 = (4, 2, 3)$, $a_2 = (2, 3, 4)$,

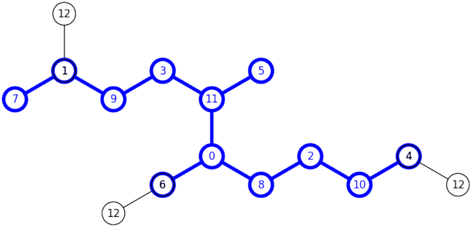


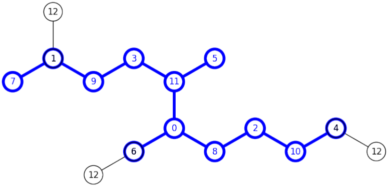
## Example : Fully Subtractive $v = (6, 8, 11)$

- Step 3 : $v_3 = (2, 2, 1)$, $a_3 = (3, 4, 6)$,




- Step 4 : $v_4 = (1, 1, 1)$, $a_4 = (6, 8, 11)$,

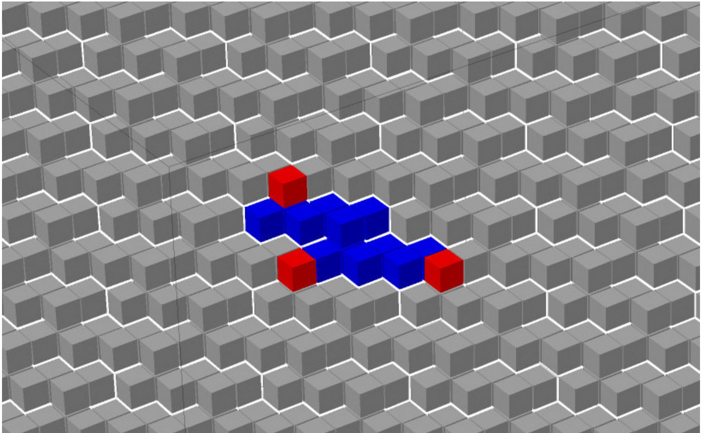





$\mathcal{P}((6, 8, 11), 13)$

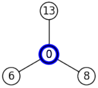
Expected properties of the pattern:

- Connected.
- Provides period vectors.
- Spans $\mathcal{P}(v, \omega)$ with these vectors.
- Should be as small as possible, to avoid redundancy.

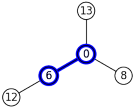


# Example, Fully Subtractive $v = (6, 8, 13)$

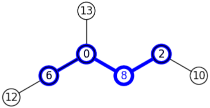
- Step 0 : $v_0 = (6, 8, 13)$, $a_0 = (1, 1, 1)$,



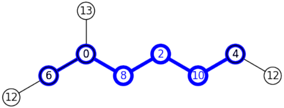

- Step 1 : $v_1 = (6, 2, 7)$, $a_1 = (1, 2, 2)$,



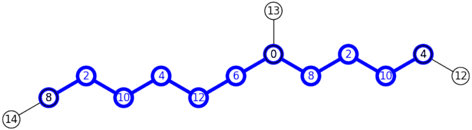

- Step 2 : $v_2 = (4, 2, 5)$, $a_2 = (2, 3, 4)$,




- Step 3 : $v_3 = (2, 2, 3)$, $a_3 = (3, 4, 6)$,




- Step 4 : $v_4 = (2, 0, 1)$, $a_4 = (5, 7, 11)$,



## Fully Subtractive

Let $v \in (\mathbb{N} \setminus \{0\})^3$ with $\gcd(v) = 1$ and $(a, b, c) = \mathrm{sort}(v)$ (i.e. $a \leq b \leq c$) :

- If $a + b \leq c$ then let $(a', b', c') = \mathrm{sort}(\mathbf{FS}(v))$ then $a' + b' \leq c'$.
- If $a = b < c$, then one coordinate of $\mathbf{FS}(v)$ is 0.

**Definition**

Let $(a, b, c) = \mathrm{sort}(v)$, the vector $v$ satisfies the condition **happy fully** if $a + b > c$ and $a \neq b$.

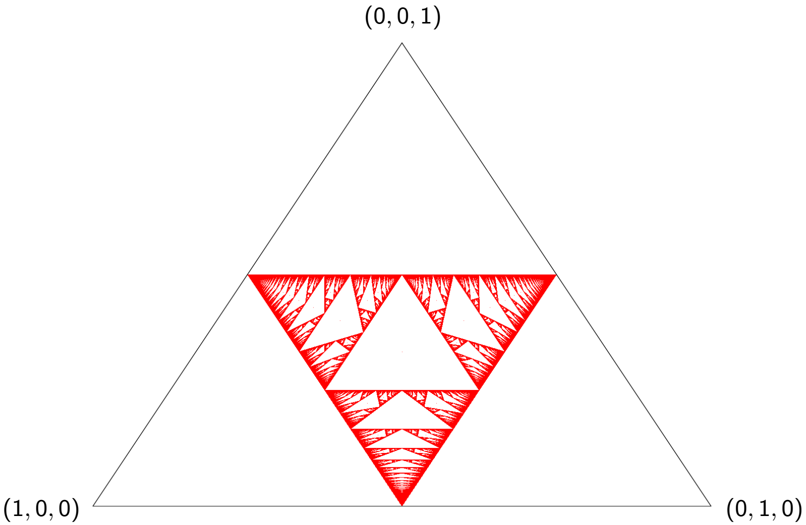
**Definition**

Let $\mathcal{K}$ be the set of vectors $v$ such $\mathbf{FS}^N(v) = (1, 1, 1)$ for some $N \geq 1$.

**Lemma**

*Let $v \in (\mathbb{N} \setminus \{0\})^3$, $v \notin \mathcal{K}$ iff there exists $n \geq 0$ such that $\mathbf{FS}^n(v)$ does not satisfy happy fully.*

$v \xrightarrow{\mathbf{FS}} \cdots \xrightarrow{\mathbf{FS}} (1,1,1)$



# New generalized continued fraction algorithms

Let **X** denote algorithm **Brun** or **Selmer**.

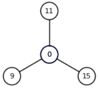
# New generalized continued fraction algorithms

Let **X** denote algorithm **Brun** or **Selmer**.

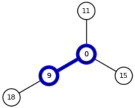
| Algorithm **FSX** |
|---|
| **Input :** $v \in \mathbb{N}^3$. |
| **If** $v$ satisfies happy fully **then**<br>    Use **FS**.<br>**else**<br>    Use **X**.<br>**end if** |

# Example using **FSB**, $v = (9, 15, 11) \notin \mathcal{K}$

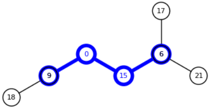
$$v_0 = (9, 15, 11) \quad a_0 = (1, 1, 1)$$



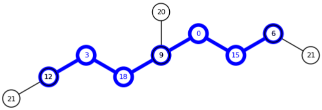

$$\xrightarrow{\textbf{FS}} \quad v_1 = (9, 6, 2) \quad a_1 = (1, 2, 2)$$



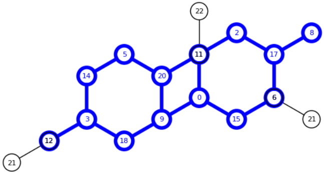

$$\xrightarrow{\textbf{Brun}} \quad v_2 = (3, 6, 2) \quad a_2 = (2, 3, 3)$$




$$\xrightarrow{\textbf{Brun}} \quad v_3 = (3, 3, 2) \quad a_3 = (3, 5, 4)$$




$$\xrightarrow{\textbf{FS}} \quad v_4 = (1, 1, 2) \quad a_4 = (6, 10, 7)$$

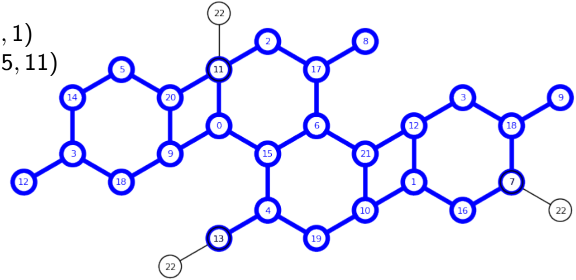


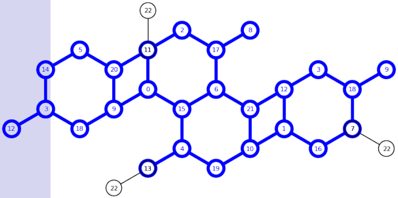
### Theorem

*Using the algorithm* ***FSB*** *or* ***FSS****, for all vector* $v \in (\mathbb{N} \setminus \{0\})^3$ *with* $\gcd(v) = 1$*,*

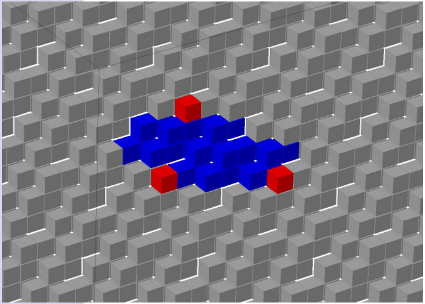
1. $\exists N$ *such that* $v_N = (1,1,1)$*.*
2. *Vectors of* $L_N$ *have same height, providing period vectors.*
3. $B_N \cup L_N$ *is connected.*
4. $B_N \cup L_N$ *spans* $\mathcal{P}(v, \omega)$ *with* $\frac{\|v\|_1}{2} \leq \omega < \|v\|_1$*.*

$\xrightarrow{\textbf{Brun}}$ $v_5 = (1,1,1)$
$a_5 = (9,15,11)$






$\mathcal{P}((9, 15, 11), 23)$



# Conclusion

Good:

- Build a pattern that spans a digital plane for any rational normal vector.
- Construction is recursive and based on continued fractions algorithms.
- Generalizes Voss' *splitting formula* (equiv. *standard factorization* of Christoffel words) to higher dimensions.

~~Problems:~~ Open questions :

- Find a gcd algorithm that builds minimal patterns.
- Control the height of the pattern.
- Control the anisotropy of the patterns (avoid stretched forms in favor of *potato-likeness*).
- Apply recursive structure to image analysis algorithms.