Generation of digital planes using generalized continued-fractions algorithms

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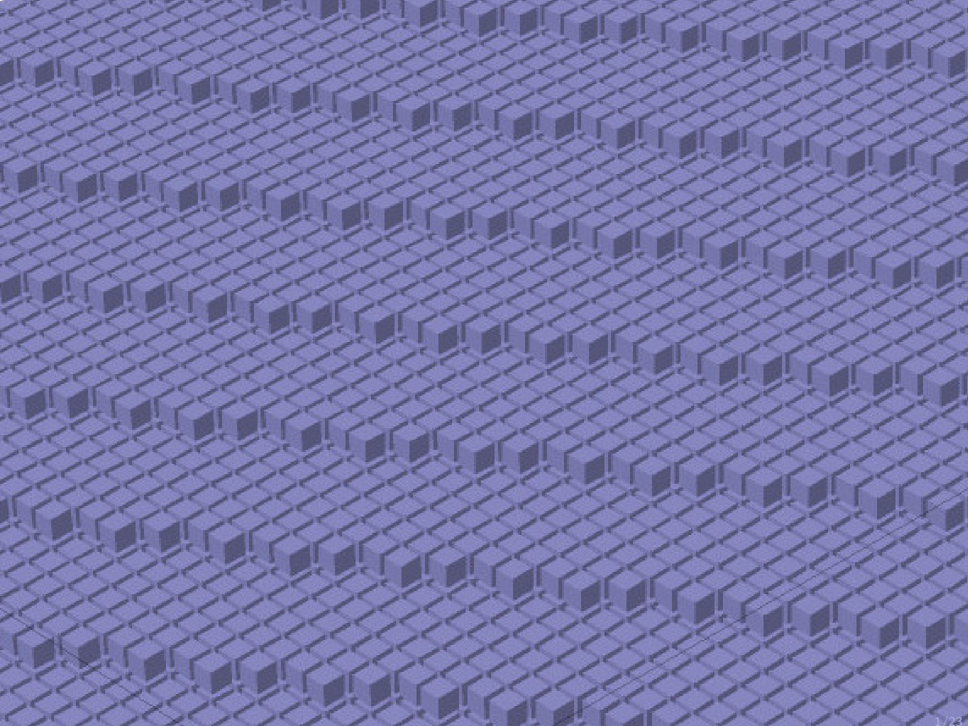


Digital lines and planes

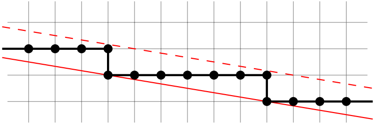
Definition ([Reveillès 91])

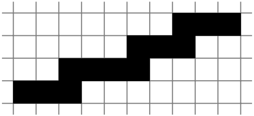
The **digital line/plane/hyperplane** $\mathcal{P}(v,\mu,\omega)$ with **normal vector** $v \in \mathbb{Z}^d$, **thickness** $\omega \in \mathbb{N}$ and **shift** $\mu \in \mathbb{R}$ is the subset of $\mathbb{Z}^d$ defined by:

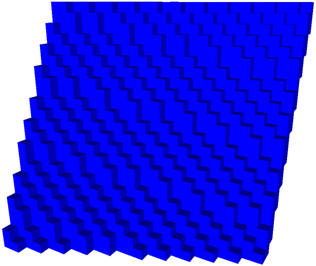
$$\mathcal{P}(v,\mu,\omega) = \left\{ x \in \mathbb{Z}^d \mid 0 \le \langle x, v\rangle - \mu < \omega \right\}$$

$\mathcal{P}((1,6),7,0)$

$0 \le 1x + 6y < 7$







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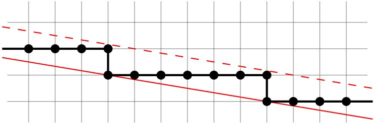
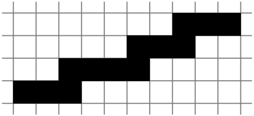
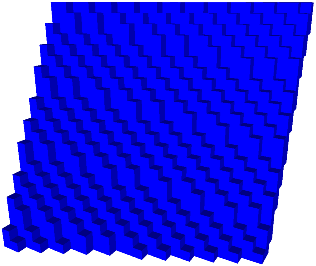
$\mathcal{P}(\ (1,6),\ 7\ )$

$0 \leq 1x + 6y < 7$



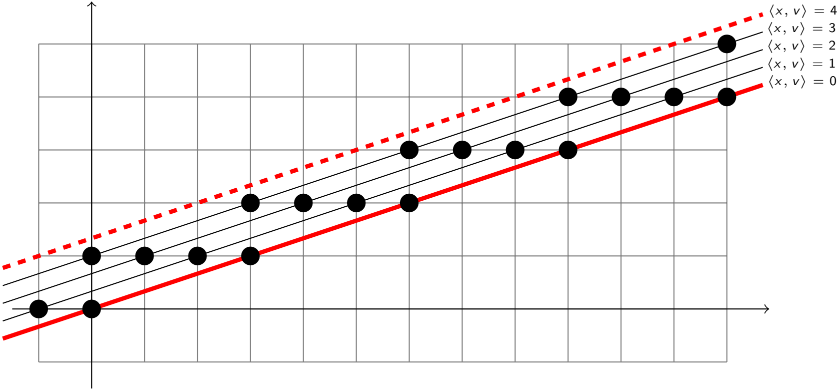




Periodic structure of a digital line

Example with $v = (-3, 1)$:

- $\langle x, v \rangle$ is the **height** of x,
- $\mathcal{P}(v, 4) = \{x \in \mathbb{Z}^2 \mid 0 \leq \langle x, v \rangle < 4\}$.



Periodic structure
Construction guided by Euclid
Using Fully Subtractive
New algorithms

Periodic structure of a digital line

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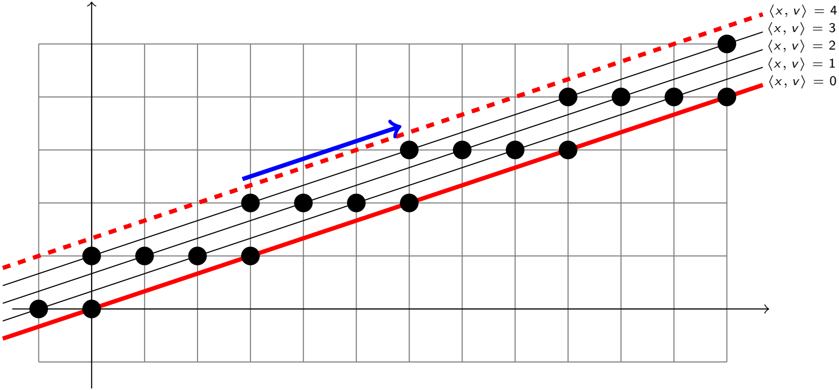
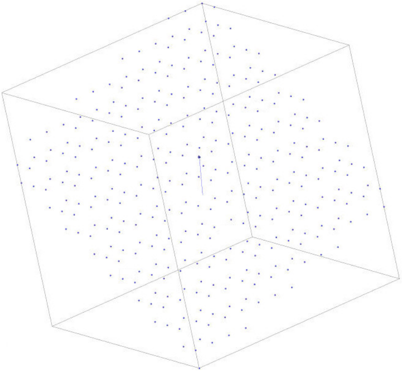
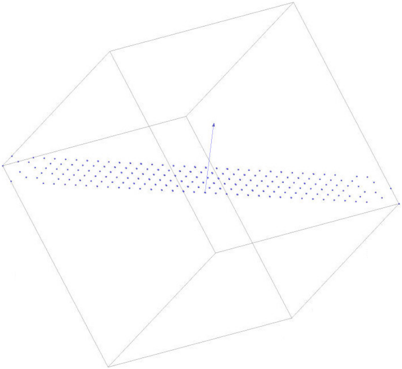


- $\langle x, v\rangle = \langle y, v\rangle \implies y - x$ is a period vector.
- A point of each height from 0 to $\|v\|_1 - 1$ appear in a period.

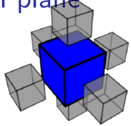
Periodic structure of a digital plane

$$v = (1,2,3), \quad \mathcal{P}(v,6) = \{x \in \mathbb{Z}^3 \mid 0 \leq \langle x, v\rangle < 6\}$$





Periodic structure of a digital plane



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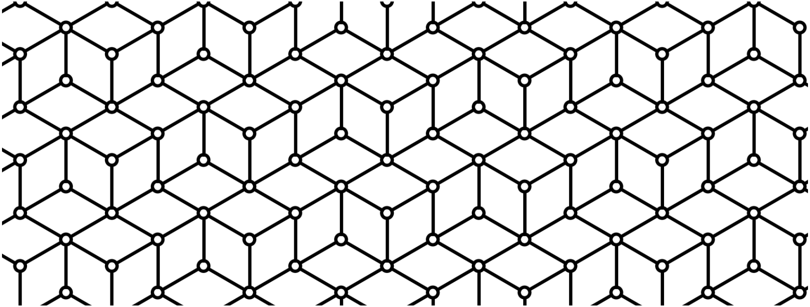
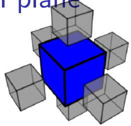


Periodic structure of a digital plane



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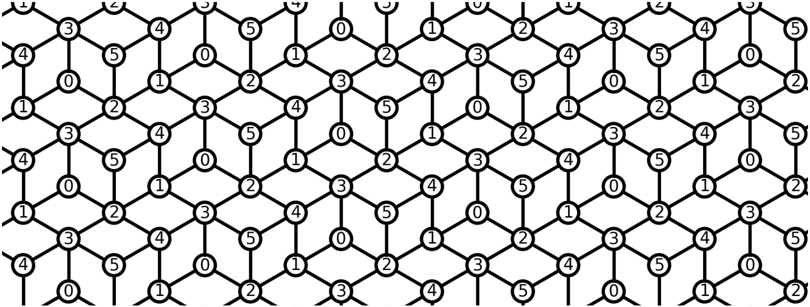

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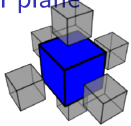

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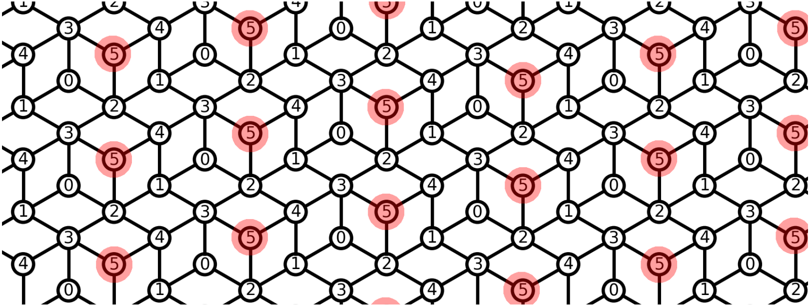


- $\langle x, v \rangle = \langle y, v \rangle \implies y - x$ is a period vector.

Periodic structure of a digital line

Definition

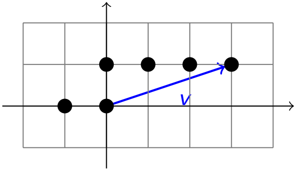
A set of points $S \subset \mathbb{Z}^d$ provided with a set of vectors $(b_i)_{i=1}^n \in \mathbb{Z}^d$ **spans** an infinite set $\Omega \subset \mathbb{Z}^d$ if

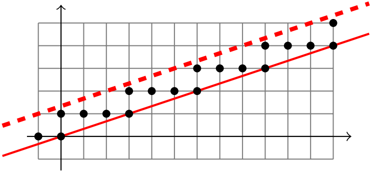
$$\Omega = \bigcup_{x \in \mathbb{Z}b_1 + \mathbb{Z}b_2 + \ldots + \mathbb{Z}b_n} (S + x).$$

(Like a tiling without a disjoint union.)

Example :

The set :

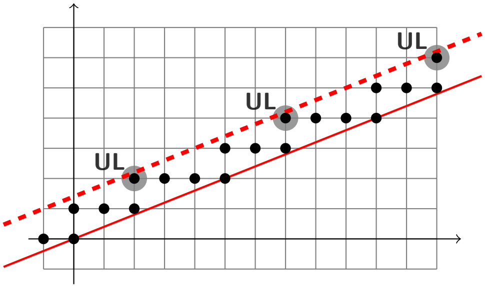




provided with vector $v = (3, 1)$ spans the digital line $\mathcal{P}((-3, 1), 4)$.

Main pattern of a digital line

- A point $x \in \mathcal{P}(v, \|v\|_1)$ is a **upper leaning point**, noted **UL**, if its height, $\langle x, v \rangle$ is maximal.



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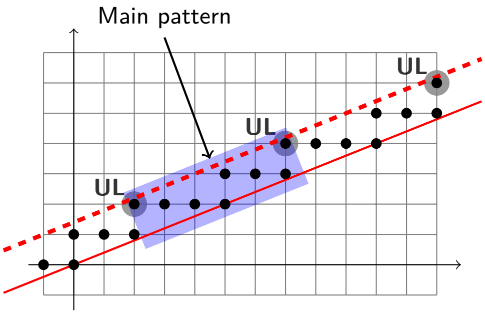
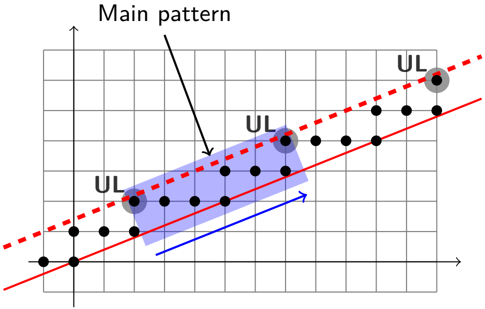
- A point $x \in \mathcal{P}(v, \|v\|_1)$ is a **upper leaning point**, noted **UL**, if its height, $\langle x, v\rangle$ is maximal.
- The **main pattern** of a digital line is a set of points bounded by two consecutive upper leaning points.



Main pattern of a digital line

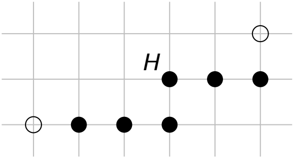
- A point $x \in \mathcal{P}(v, \|v\|_1)$ is a **upper leaning point**, noted **UL**, if its height, $\langle x, v \rangle$ is maximal.
- The **main pattern** of a digital line is a set of points bounded by two consecutive upper leaning points.
- Let v be the vector defined by two consecutive **UL**, a main pattern provided with v spans its digital line.



Main pattern of a digital line

- ○ : upper leaning points.
- Let H be the highest point among {●}, a **Bezout** point.

Main pattern for slope 2/5.



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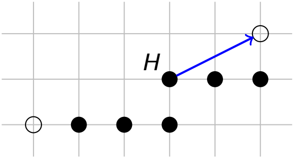

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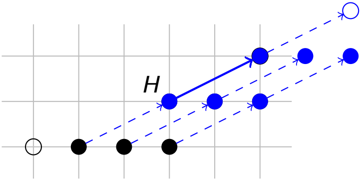

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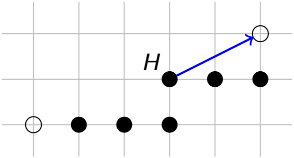
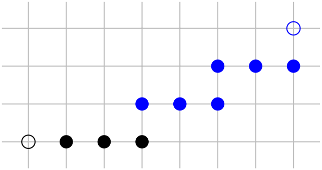


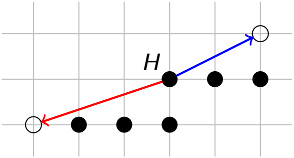


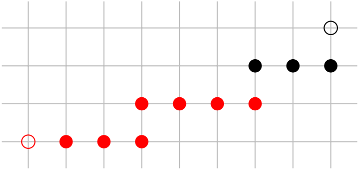
Main pattern for slope 3/8.

Main pattern of a digital line

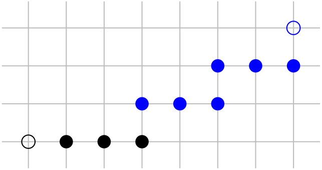
- ○ : upper leaning points.
- Let H be the highest point among {●}, a **Bezout** point.

Main pattern for slope 2/5.



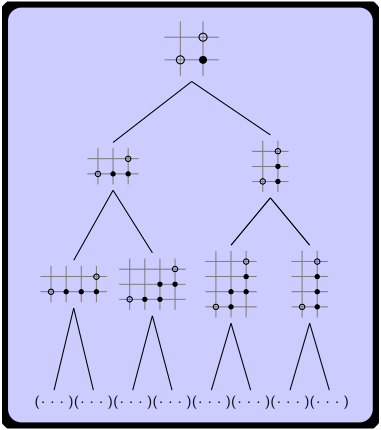


Main pattern for slope 3/7.

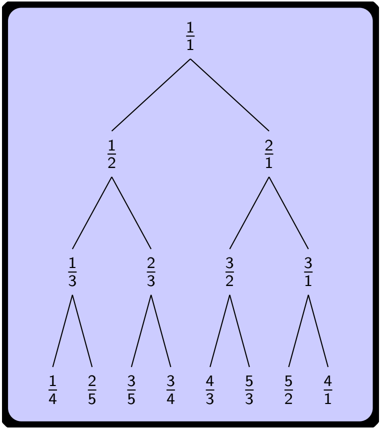


Main pattern for slope 3/8.

Stern-Brocot Tree



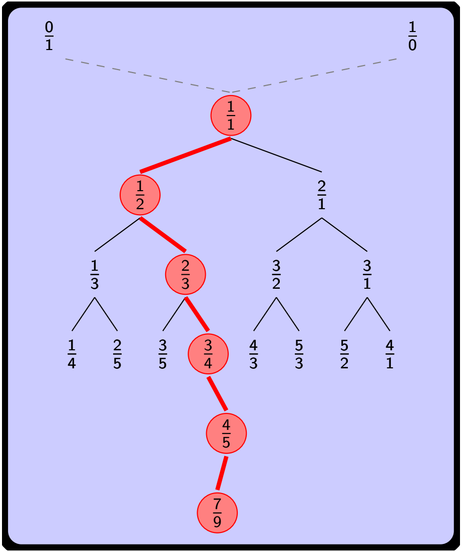
Stern-Brocot tree.



Every irreducible fraction appears exactly once in the Stern-Brocot tree.

Euclid Algorithm

Stern-Brocot tree



Euclid algorithm	Approximation
(7, 9)	(1, 1)
↓	↓
(7, 2)	(1, 2)
↓	↓
(5, 2)	(2, 3)
↓	↓
(3, 2)	(3, 4)
↓	↓
(1, 2)	(4, 5)
↓	↓
(1, 1)	(7, 9)

Matricial view

	Euclid algorithm	Approx.
n	v_n	a_n
0	$(7, 9)$	$(1, 1)$
	$\downarrow$	$\downarrow$
1	$(7, 2)$	$(1, 2)$
	$\downarrow$	$\downarrow$
2	$(5, 2)$	$(2, 3)$
	$\downarrow$	$\downarrow$
3	$(3, 2)$	$(3, 4)$
	$\downarrow$	$\downarrow$
4	$(1, 2)$	$(4, 5)$
	$\downarrow$	$\downarrow$
5	$(1, 1)$	$(7, 9)$

Euclid algorithm
Given a vector (x, y), return

- $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ if $x < y$,
- $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ if $x > y$,
- **stop** if $x = y$.

Given a vector $v \in (\mathbb{N} \setminus \{0\})^2$, let :

- $v_0 = v$,
- For all $n \geq 1$: $\begin{cases} M_n = \textbf{Euclid}(v_{n-1}) \\ v_n = M_n v_{n-1}. \end{cases}$

Matricial view

	Euclid algorithm	Approx.
n	v_n	a_n
0	$(\underline{7}, 9)$	$(1, 1)$
	$\downarrow$	$\downarrow$
1	$(7, \underline{2})$	$(1, 2)$
	$\downarrow$	$\downarrow$
2	$(5, \underline{2})$	$(2, 3)$
	$\downarrow$	$\downarrow$
3	$(3, \underline{2})$	$(3, 4)$
	$\downarrow$	$\downarrow$
4	$(\underline{1}, 2)$	$(4, 5)$
	$\downarrow$	$\downarrow$
5	$(1, 1)$	$(7, 9)$

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Property

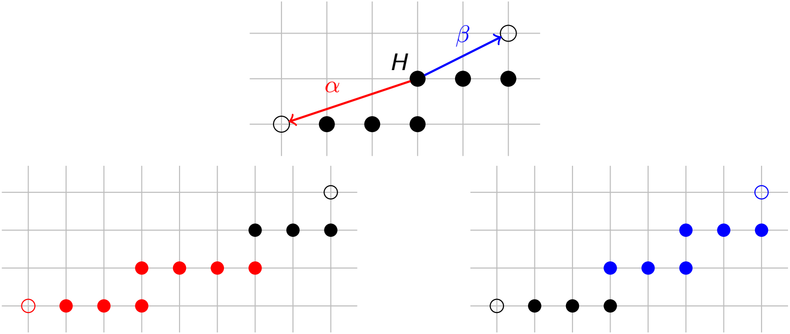
- $v_n = M_n M_{n-1} \cdots M_1 v$
- $a_n = M_1^{-1} M_2^{-1} \cdots M_n^{-1} \binom{1}{1}$

Matricial view

Let UL_0 and UL_1 be two upper leaning points of a main pattern of $\mathcal{P}(a_n, \|a_n\|_1)$ and H be the Bezout point. Let $\alpha = UL_0 - H$ and $\beta = UL_1 - H$, then

$$M_1^\top M_2^\top \cdots M_n^\top = \left[\begin{array}{cc} \alpha & \beta \end{array} \right]$$

$$M_1^\top \cdots M_n^\top e_1 = \alpha, \qquad M_1^\top \cdots M_n^\top e_2 = \beta.$$



The Translation-Union Construction

Construction
[Domenjoud,Vuillon 12],
[Berthé, Jamet, Jolivet, P. 2013]

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

$v_n = M_n v_{n-1}$

δ_n : the index of the coordinate of v_{n-1} that is subtracted,

$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}$, *(translation)*

$B_n = B_{n-1} \cup (T_n + B_{n-1})$, *(body)*

$H_n = \sum_{i \in \{1,\ldots,n\}} T_i$, *(highest point)*

$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}$. *(legs)*

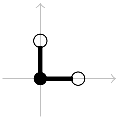
Note that:

$H_n \in B_n$,

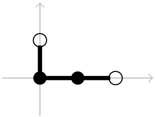
$L_n \cap B_n = \emptyset$.

$\bullet \in B_n$, $\circ \in L_n$

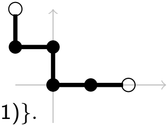
$v_0 = (2,3)$,
$a_0 = (1,1)$
$H_0 = (0,0)$,
$L_0 = \{(1,0),(0,1)\}$.



$v_1 = (2,1), \delta_1 = 1$
$a_1 = (1,2)$
$T_1 = (1,0)$
$H_1 = (1,0)$,
$L_1 = \{(2,0),(0,1)\}$.



$v_2 = (1,1), \delta_2 = 2$
$a_2 = (2,3)$
$T_2 = (-1,1)$
$H_2 = (0,1)$,
$L_2 = \{(2,-1),(-1,1)\}$.



3D continued fraction algorithms

Euclid algorithm : given two number subtract the smaller to the larger.
$(7,9) \to (7,2) \to (5,2) \to (3,2) \to (1,2) \to (1,1) \to (1,0)\circlearrowleft$

3D continued fraction algorithms

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$(7,9) \to (7,2) \to (5,2) \to (3,2) \to (1,2) \to (1,1) \to (1,0)\circlearrowleft$

Given three numbers :

- **Selmer** : subtract the smallest to the largest.
  $(3,7,5) \to (3,4,5) \to (3,4,2) \to (3,2,2) \to (1,2,2) \to (1,2,0)\circlearrowleft$.
- **Brun** : subtract the second largest to the largest.
  $(3,7,5) \to (3,2,5) \to (3,2,2) \to (1,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)\circlearrowleft$.
- **Fully subtractive** : subtract the smallest to the two others.
  $(3,7,5) \to (3,4,2) \to (1,2,2) \to (1,1,1) \to (1,0,0)\circlearrowleft$.
- **Poincaré** : subtract the smallest to the mid and the mid to the largest.
  $(3,7,5) \to (3,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)\circlearrowleft$.
- **Arnoux-Rauzy** : subtract the sum of the two smallest to the largest (not always possible).
  $(3,7,5) \to \text{impossible.}$
- ...

Example : Fully Subtractive $v = (6, 8, 11)$

Construction

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

$v_n = M_n v_{n-1}$

δ_n : the index of the coordinate of v_{n-1} that is subtracted,

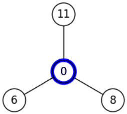
$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}$, *(translation)*

$B_n = B_{n-1} \cup (T_n + B_{n-1})$, *(body)*

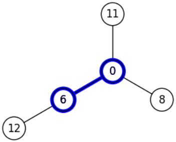
$H_n = \sum_{i \in \{1,\dots,n\}} T_i$, *(highest point)*

$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}$. *(legs)*

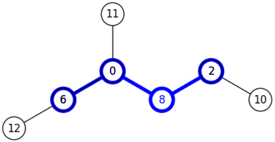
- Step 0 : $v_0 = (6, 8, 11)$, $a_0 = (1, 1, 1)$,



- Step 1 : $v_1 = (6, 2, 5)$, $a_1 = (1, 2, 2)$,

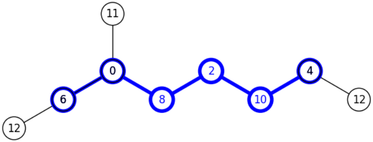


- Step 2 : $v_2 = (4, 2, 3)$, $a_2 = (2, 3, 4)$,

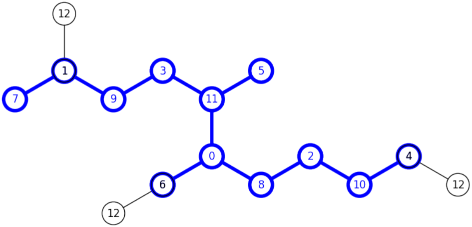


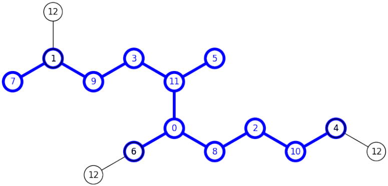
Example : Fully Subtractive $v = (6, 8, 11)$

- Step 3 : $v_3 = (2, 2, 1)$, $a_3 = (3, 4, 6)$,



- Step 4 : $v_4 = (1, 1, 1)$, $a_4 = (6, 8, 11)$,

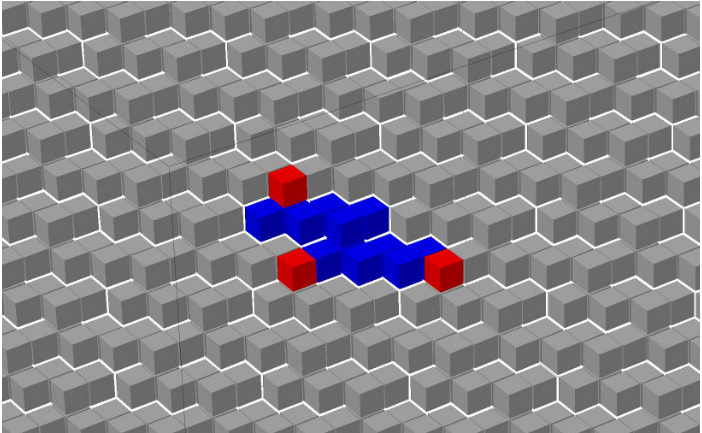




$\mathcal{P}((6, 8, 11), 13)$

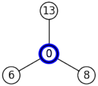
Expected properties of the pattern:

- Connected.
- Provides period vectors.
- Spans $\mathcal{P}(v, \omega)$ with these vectors.
- Should be as small as possible, to avoid redundancy.

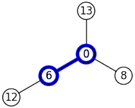


Example, Fully Subtractive $v = (6, 8, 13)$

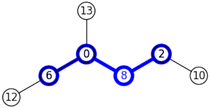
- Step 0 : $v_0 = (6, 8, 13),\ a_0 = (1, 1, 1),$



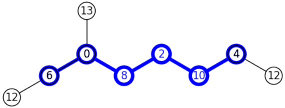
- Step 1 : $v_1 = (6, 2, 7),\ a_1 = (1, 2, 2),$



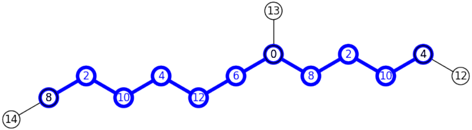
- Step 2 : $v_2 = (4, 2, 5),\ a_2 = (2, 3, 4),$



- Step 3 : $v_3 = (2, 2, 3),\ a_3 = (3, 4, 6),$



- Step 4 : $v_4 = (2, 0, 1),\ a_4 = (5, 7, 11),$



Fully Subtractive

Definition

Let $\mathcal{K}$ be the set of vectors v such $\mathbf{FS}^N(v) = (1,1,1)$ for some $N \geq 1$.

Let $v \in (\mathbb{N} \setminus \{0\})^3$ with $\gcd(v) = 1$ and $(a,b,c) = \text{sort}(v)$ (i.e. $a \leq b \leq c$), two conditions:

(1) If $a + b \leq c$ then let $(a', b', c') = \text{sort}(\mathbf{FS}(v))$ then $a' + b' \leq c'$.

Example : $(2,3,6) \xrightarrow{\mathbf{FS}} (2,1,4) \xrightarrow{\mathbf{FS}} (1,1,3) \xrightarrow{\mathbf{FS}} (1,0,2)$.

(2) If $a = b < c$, then one coordinate of $\mathbf{FS}(v)$ is 0.

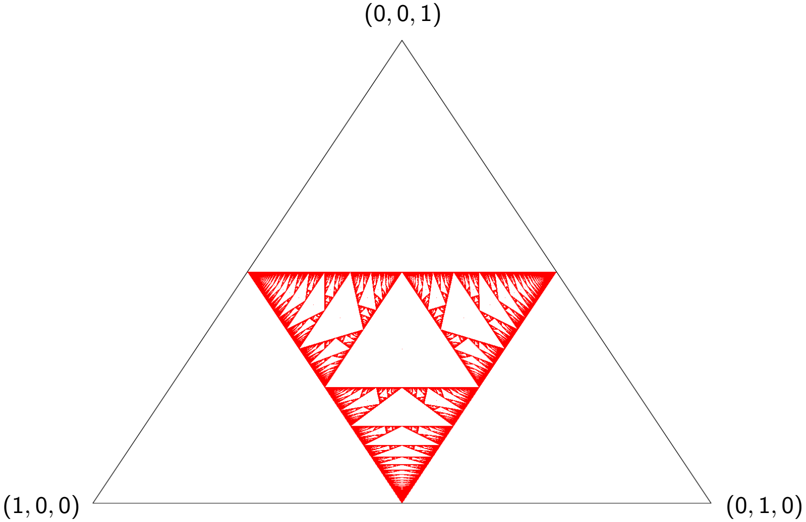
Example : $(2,2,3) \xrightarrow{\mathbf{FS}} (2,0,1)$.

Lemma

Let $v \in (\mathbb{N} \setminus \{0\})^3$, $v \notin \mathcal{K}$ iff there exist $n \geq 0$ such that $\mathbf{FS}^n(v)$ satisfies condition (1) or (2).

The set $\mathcal{K}$

$$v \xrightarrow{\textbf{FS}} \cdots \xrightarrow{\textbf{FS}} (1,1,1)$$



New generalized continued fraction algorithms

Idea : If the vector *looks good*, use **FS**, otherwise use some thing else. . . like **Brun** or **Selmer**.

New generalized continued fraction algorithms

Idea : If the vector *looks good*, use **FS**, otherwise use some thing else. . . like **Brun** or **Selmer**.

Algorithm **FSB**
Input : $v \in \mathbb{N}^3$.

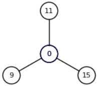
If v satisfies (1) or (2) **then**
Use **Brun**.
else
Use **FS**.
end if

Algorithm **FSS**
Input : $v \in \mathbb{N}^3$.

If v satisfies (1) or (2) **then**
Use **Selmer**.
else
Use **FS**.
end if

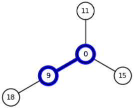
Example using **FSB**, $v = (9, 15, 11) \notin \mathcal{K}$

$$v_0 = (9, 15, 11)$$
$$a_0 = (1, 1, 1)$$



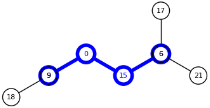
$\xrightarrow{\textbf{FS}}$

$$v_1 = (9, 6, 2)$$
$$a_1 = (1, 2, 2)$$



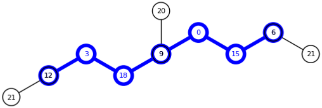
$\xrightarrow{\textbf{Brun}}$

$$v_2 = (3, 6, 2)$$
$$a_2 = (2, 3, 3)$$



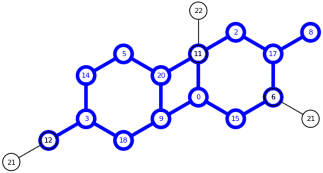
$\xrightarrow{\textbf{Brun}}$

$$v_3 = (3, 3, 2)$$
$$a_3 = (3, 5, 4)$$



$\xrightarrow{\textbf{FS}}$

$$v_4 = (1, 1, 2)$$
$$a_4 = (6, 10, 7)$$

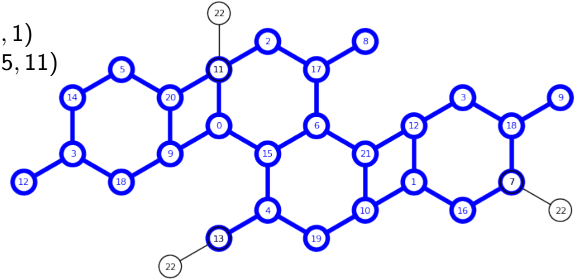


Theorem

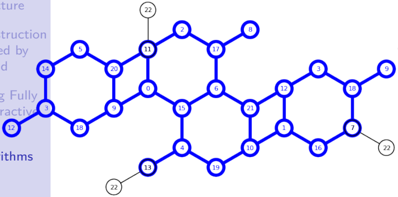
*Using the algorithm **FSB** or **FSS**, for all vector $v \in (\mathbb{N} \setminus \{0\})^3$ with* $\gcd(v) = 1$,

1. $\exists N$ *such that* $v_N = (1, 1, 1)$.
2. *Vectors of* L_N *have same height, providing period vectors.*
3. $B_N \cup L_N$ *is connected.*
4. $B_N \cup L_N$ *spans* $\mathcal{P}(v, \omega)$ *with* $\frac{\|v\|_1}{2} \leq \omega \leq \|v\|_1$.

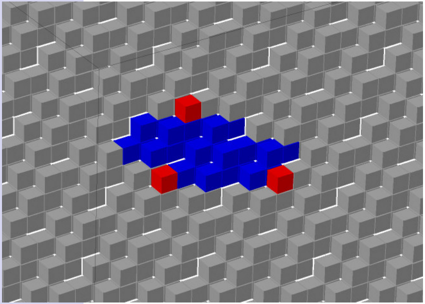
$\xrightarrow{\textbf{Brun}}$ $v_5 = (1, 1, 1)$
$a_5 = (9, 15, 11)$



Conclusion



$\mathcal{P}((9, 15, 11), 23)$



Good:

- Build a pattern that spans a digital plane for any rational normal vector.
- Construction is recursive and based on continued fractions algorithms.
- Generalizes Voss' *splitting formula* (equiv. *standard factorization* of Christoffel words) to higher dimensions.

~~Problems:~~ Open questions :

- Find a gcd algorithm that builds minimal patterns.
- Control the height of the pattern.
- Control the anisotropy of the patterns (avoid stretched forms in favor of *potato-likeness*).
- Apply recursive structure to image analysis algorithms.