Une approche "aussi locale que possible" pour calculer la normale d'un plan discret, approche basée sur un algorithme de fractions continues ad-hoc mélangeant Fully subtractive, Brun, Selmer et des versions généralisées de ceux-ci

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Université de Savoie

Réunion DYNA3S, Nancy le 22 sept. 2014

Plan :

Droites et plans dicrets
Le problème du calcul de la normale
Structure d'une droite discrète et la triangulation de Delaunay
Ze algorithm
Version arithmétique
Variante pour obtenir une base réduite

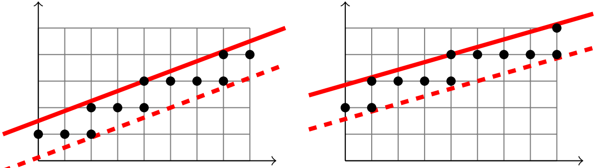
Arithmetic DSL

Definition (Reveillès (1991), Kovalev (1990))

The *standard arithmetic digital straight line* is :

$$D((a,b),\mu) = \{(x,y) \in \mathbb{Z}^2 \mid 0 \leq ax - by + \mu < |a| + |b|\}$$

- b/a is the *slope*,
- μ is the *shift*.

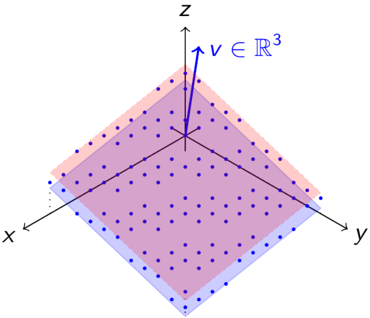


$D(3,8,12)$ $D(2,7,20)$

A standard DSL forms a connected path without loops.

A finite and connected part of a DSS is called a Digital Straight Segment (DSS).

Arithmetic discrete hyperplanes



Definition (Forchhammer 89 and Reveillès 91)

Discrete hyperplane with normal vector $\boldsymbol{v} \in \mathbb{R}^3 \setminus \{0\}$, shift $\mu \in \mathbb{R}$, thickness ω.

$$\mathcal{P}(\boldsymbol{v}, \mu) = \{\boldsymbol{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \boldsymbol{v}, \boldsymbol{x} \rangle + \mu < \omega\}$$

When $\omega = \|v\|_1$ the plane is called standard.

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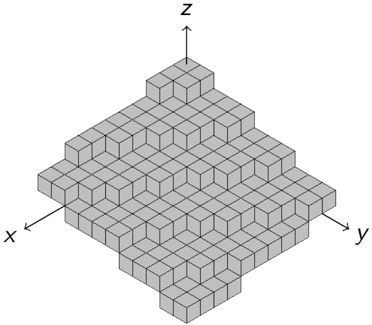

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Calcul de la normale

Étant donné un prédicat répondant à la question :

“Est-ce que que le point entier $x \in \mathcal{P}(\mathbf{N}, \mu, \omega)$?”

On souhaite décrire $\mathcal{P}(\mathbf{N}, \mu, \omega)$:

- Valeur de $\mathbf{N}$, μ et ω,
- Un point d'appui sup. (i.e. $\langle x, \mathbf{N} \rangle = \omega - 1$),
- Un point d'appui inf. (i.e. $\langle x, \mathbf{N} \rangle = 0$),
- Base (réduite) du lattice (i.e. $\langle x, \mathbf{N} \rangle = 0$),
- Vecteur de *passage* (Bezout) (i.e. x tq $\langle x, \mathbf{N} \rangle = 1$).

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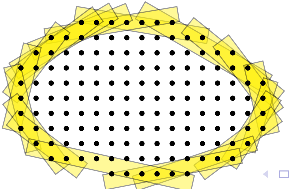
- On souhaite une approche "aussi locale que possible".

Definition (Tangential cover (Feschet, Tougne 1999))

The *tangential cover* of a discrete shape is the sequence of all maximal DSS on its boundary.



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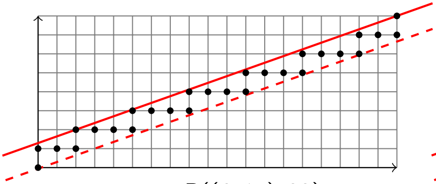
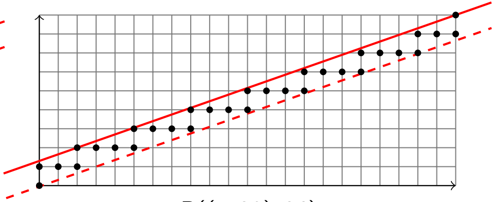


$D((6, 17), 22)$



$D((7, 20), 26)$

Calcul de la normale

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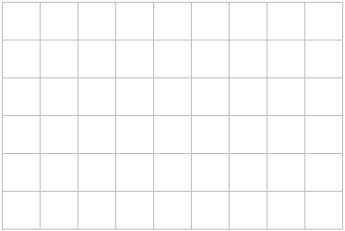
- On souhaite une approche "aussi locale que possible".
- Sachant très bien que c'est impossible. . .
- Hypothèse :
  - $\mathbf{N} \in (\mathbb{N} \setminus \{0\})^3$.
  - $\gcd(\mathbf{N}) = 1$.
  - $\|\mathbf{N}\|_\infty$ est borné par. . . disons `MAXBOUND`.

Christoffel words

Definition (Christoffel Word)

A **Christoffel word** codes digital path right below a segments between two consecutive interger points



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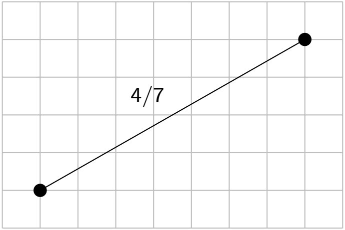

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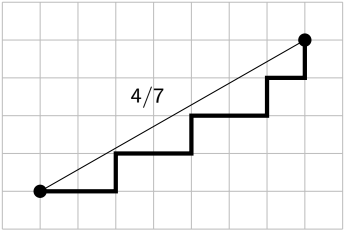

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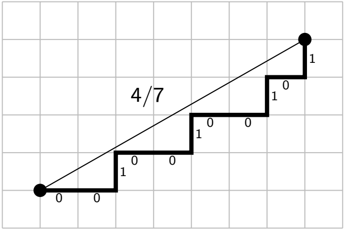

$w = 00100100101$ is the Christoffel word of **slope** $4/7$.

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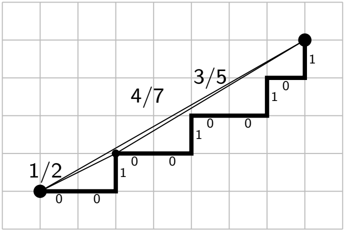

$w = 001 \cdot 00100101$ is the Christoffel word of **slope** $4/7$.

Theorem (Borel, Laubie, 1993)

Any Christoffel word, other than 0 *or* 1*, can be written in a unique way as a product of two Christoffel words.*

This is called the **standard factorization**, noted $w = (u, v)$.

Christoffel Tree

If (u, v) is a standard factorization, then (u, uv) and (uv, v) are standard factorizations of Christoffel words.

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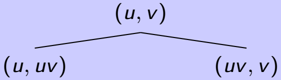
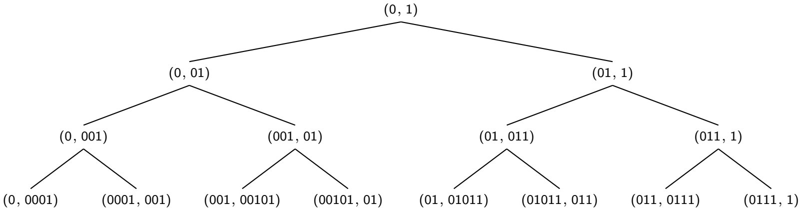
The **Christoffel Tree** is the tree obtained, starting from $(0, 1)$, using the rule :





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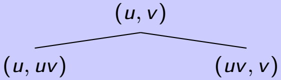
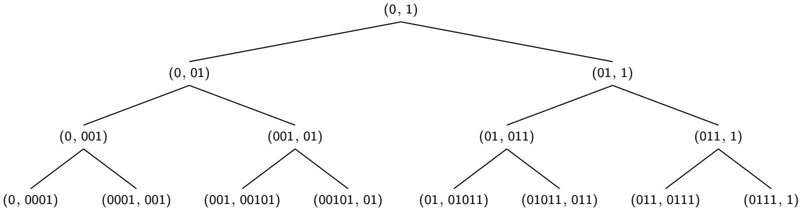


Theorem

Every Christoffel word appears exactly once in the Christoffel Tree.



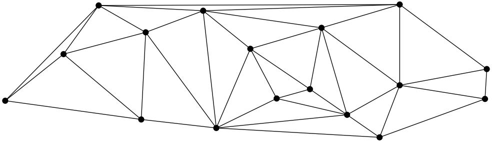
Delaunay triangulation

Definition (Triangulation of a finite set of points $\mathcal{S}$)

Partition of the convex hull of $\mathcal{S}$ into triangular facets, whose vertices are points of $\mathcal{S}$.

Definition (Delaunay condition)

The interior of the circumcircle of each triangular facet does not contain any set point.



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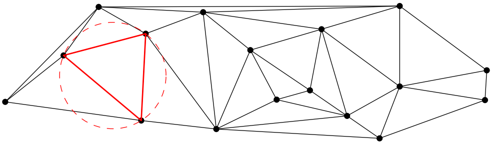

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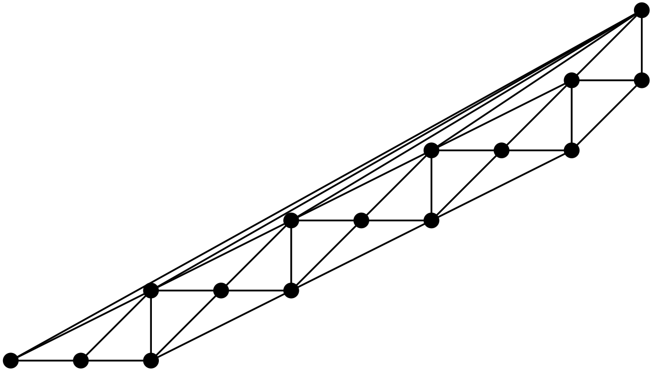
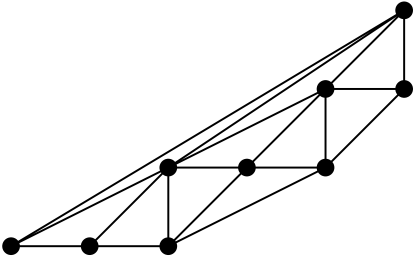
Delaunay triangulation of Christoffel words

Slope 5/9



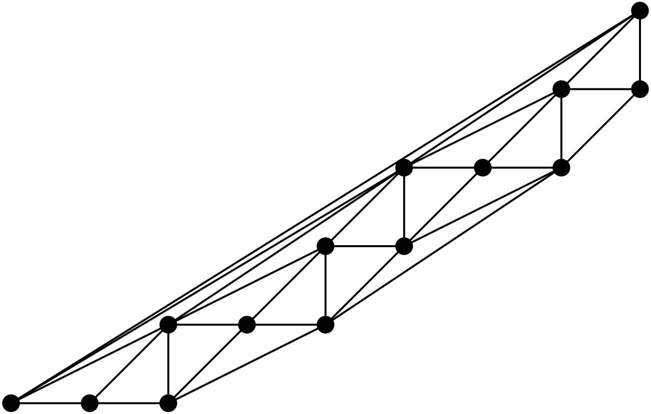
Delaunay triangulation of Christoffel words

Slope 3/5



Delaunay triangulation of Christoffel words

Slope 5/8



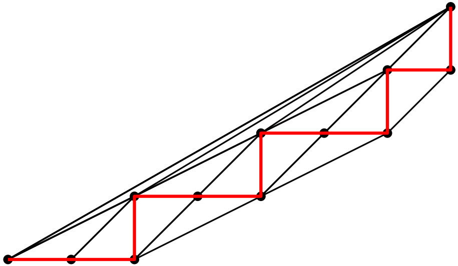
Three remarks

Theorem ([Lachaud, Roussillon 11])

La triangulation de Delaunay des points d'un mot de Christoffel contient :

- *le chemin de Christoffel,*

Christoffel de pente 4/7



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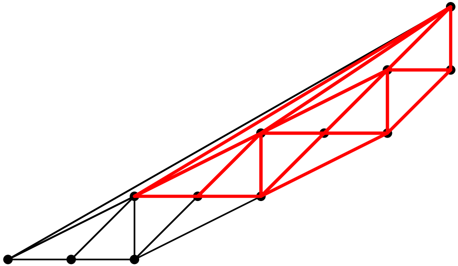

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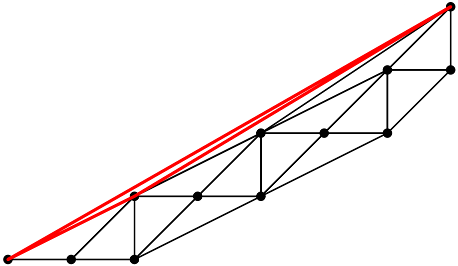
- *la factorisation standard,*

Christoffel de pente 4/7



Élément de base de l'algorithme

Notations:

- $\mathbf{N}$ est le vecteur normal au plan. (C'est lui qu'on cherche !)
- For any vector $\mathbf{v} \in \mathbb{R}^3$, $\overline{\mathbf{v}} = \langle \mathbf{v}, \mathbf{N} \rangle$,
- $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$.

Definition (Système)

- Un système est un quadruplet $\mathfrak{S} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{o}) \in (\mathbb{Z}^3)^4$.
- Un système est dit valide si :
  - $\mathbf{o}, \mathbf{o} + \mathbf{u}, \mathbf{o} + \mathbf{v}, \mathbf{o} + \mathbf{w} \in P$,
  - $\overline{\mathbf{u}}, \overline{\mathbf{v}}, \overline{\mathbf{w}} > 0$,
  - $\begin{vmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{vmatrix} = 1$,
- La normale du système $\mathfrak{S}$ est le vecteur $\hat{\mathbf{N}}(\mathfrak{S}) := (\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})$

- Initialisation : $\mathfrak{S} = (e_1, e_2, e_3, \mathbf{o})$. Il faut pour cela trouver un "coin" où se positionner pour commencer.
- Objectif : obtenir un système valide tel que $\overline{\mathbf{u}} = \overline{\mathbf{v}} = \overline{\mathbf{w}} = 1$ et $\overline{\mathbf{o}} = \omega - 2$.

Opérations

Definition

Une opération est une fonction $\lambda : (\mathbb{Z}^3)^4 \to (\mathbb{Z}^3)^4$ telle qu'étant donné $(\mathbf{u}', \mathbf{v}', \mathbf{w}', \mathbf{o}') = \lambda((\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{o}))$, il existe une matrice M_λ satisfaisant :

$$\begin{bmatrix} \mathbf{u}' \\ \mathbf{v}' \\ \mathbf{w}' \end{bmatrix} = M_\lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

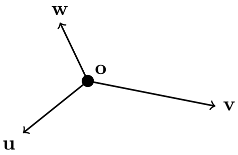
Definition

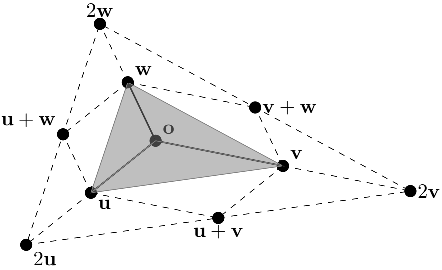
Une opération λ est valide sur un système valide $\mathfrak{S} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{o})$ si

- $(\mathbf{u}', \mathbf{v}', \mathbf{w}', \mathbf{o}') = \lambda(\mathfrak{S})$ est un système valide,
- $\overline{\mathbf{o}}' > \overline{\mathbf{o}}$,

Opérations locale

Ne considèrent que les 6 points $2\mathbf{u}, 2\mathbf{v}, 2\mathbf{w}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}$.





(abus de notation : $2\mathbf{u} \longrightarrow \mathbf{o} + 2\mathbf{u}$)

Toutes les opérations sont exprimées par rapport à une permutation σ des vecteurs $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

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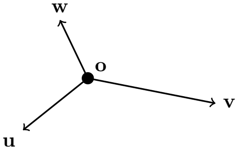

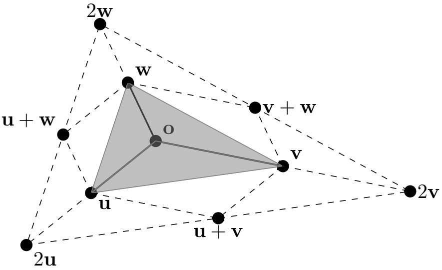

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Lemme

Étant donné $\mathbf{o} \in P$ *et deux vecteurs* $\mathbf{x}, \mathbf{y}$ *tel que* $\overline{\mathbf{x}} > 0$ *et* $\overline{\mathbf{y}} > 0$,

$$\mathbf{o} + \mathbf{x} \in P \text{ et } \mathbf{o} + \mathbf{y} \notin P \implies$$

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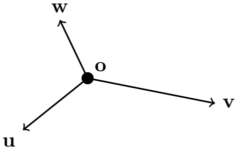

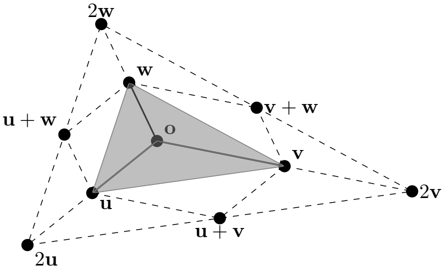

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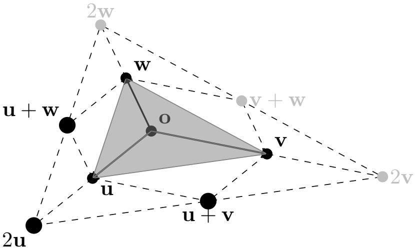
Translation

- Préconditions pour T_{Id} :
  - $\{\mathbf{o}+2\mathbf{u}, \mathbf{o}+\mathbf{u}+\mathbf{v}, \mathbf{o}+\mathbf{u}+\mathbf{w}\} \in P$.

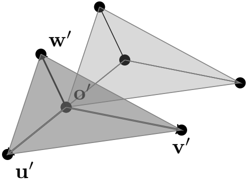
- Opération T_{Id} :

$$\begin{aligned} \mathbf{o}' &= \mathbf{o}+\mathbf{u} \\ \mathbf{u}' &= \mathbf{u} \\ \mathbf{v}' &= \mathbf{v} \\ \mathbf{w}' &= \mathbf{w} \end{aligned}$$

$$M_{T_{\mathrm{Id}}} := Id$$



$\xrightarrow{T_{\mathrm{Id}}}$



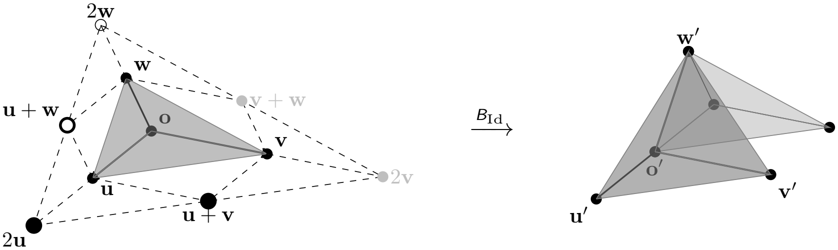
Lemme

T_σ est valide sur un système satisfaisant ses préconditions.

Brun

- Préconditions pour B_{Id} :
  - $\{\mathbf{o}+2\mathbf{u}, \mathbf{o}+\mathbf{u}+\mathbf{v}\} \in P$
  - $\{\mathbf{o}+\mathbf{u}+\mathbf{w}\} \notin P$.
- Opération B_{Id} :

$$\begin{aligned} \mathbf{o}' &= \mathbf{o}+\mathbf{u} \\ \mathbf{u}' &= \mathbf{u} \\ \mathbf{v}' &= \mathbf{v} \\ \mathbf{w}' &= \mathbf{w}-\mathbf{u} \end{aligned} \qquad M_{B_{\mathrm{Id}}} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$



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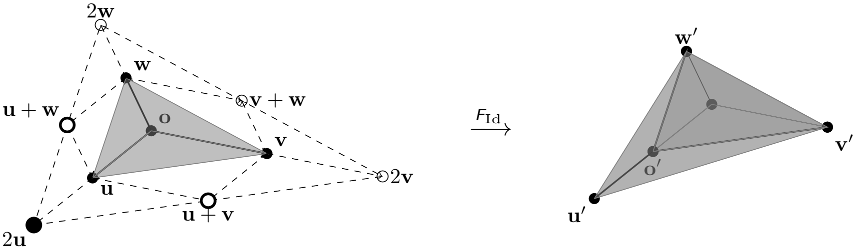
Fully subtractive

- Préconditions pour F_{Id} :
  - $\{\mathbf{o} + 2\mathbf{u}\} \in P$,
  - $\{\mathbf{o} + \mathbf{u} + \mathbf{v}, \mathbf{o} + \mathbf{u} + \mathbf{w}\} \notin P$.
- Opération F_{Id} :

$$\begin{aligned} \mathbf{o}' &= \mathbf{o} + \mathbf{w} \\ \mathbf{u}' &= \mathbf{u} \\ \mathbf{v}' &= \mathbf{v} - \mathbf{u} \\ \mathbf{w}' &= \mathbf{w} - \mathbf{u} \end{aligned} \qquad M_{F_{\mathrm{Id}}} := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$



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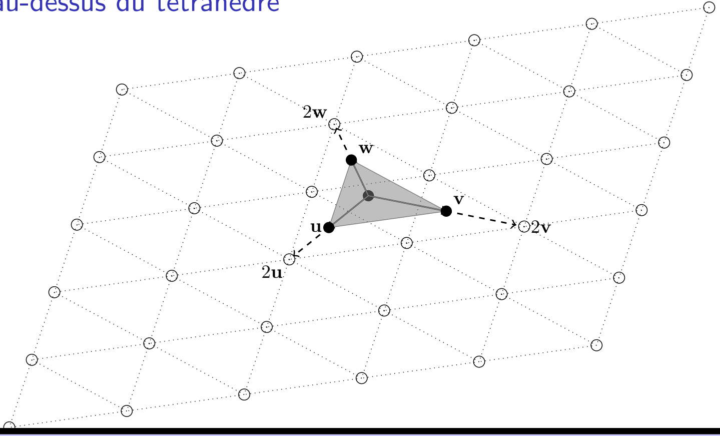
Opérations généralisées

Lemme

Si $\mathfrak{S} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{o})$ est tel que au moins un des points $\mathbf{o} + 2\mathbf{u}$, $\mathbf{o} + 2\mathbf{v}$, $\mathbf{o} + 2\mathbf{w}$ appartient à P, alors au moins une des opérations locales est valide.

- Les opération généralisées ne sont donc considérées que dans le cas où $\mathbf{o} + 2\mathbf{u}$, $\mathbf{o} + 2\mathbf{v}$, $\mathbf{o} + 2\mathbf{w}$, $\mathbf{o} + \mathbf{u} + \mathbf{v}$, $\mathbf{o} + \mathbf{u} + \mathbf{w}$, $\mathbf{o} + \mathbf{v} + \mathbf{w}$ sont tous à l'extérieur de P.

Lattice au-dessus du tétrahèdre



Definition

- $\mathbb{L} = \{\mathbf{o} + 2\mathbf{u} + \alpha(\mathbf{u} - \mathbf{v}) + \beta(\mathbf{u} - \mathbf{w}) \mid \alpha, \beta \in \mathbb{Z}\}$
- Étant donné une permutation σ :

$$\begin{array}{llcl} \mathbb{L}_\sigma : & \mathbb{Z}^2 & \to & \mathbb{Z}^3 \\ & (\alpha, \beta) & \mapsto & \mathbf{o} + 2\sigma(\mathbf{u}) + \alpha(\sigma(\mathbf{u}) - \sigma(\mathbf{v})) + \beta(\sigma(\mathbf{u}) - \sigma(\mathbf{w})) \end{array}$$

Notation, $\overline{\mathbb{L}}_\sigma(\alpha, \beta) = \overline{\mathbb{L}_\sigma(\alpha, \beta)}$.

Fully subtractive généralisé

Opération définie pour une paire $\alpha, \beta \in \mathbb{N}_+$ tels que $\alpha + \beta \geq 1$.

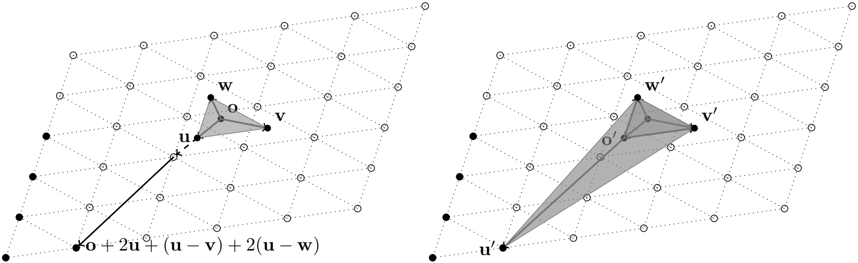
- Préconditions pour $F_{\mathrm{Id}}^{\alpha,\beta}$:
  - $\mathbf{o} + 2\mathbf{u}, \mathbf{o} + 2\mathbf{v}, \mathbf{o} + 2\mathbf{w} \notin P$,
  - $\overline{\mathbf{u}} < \overline{\mathbf{v}}$ et $\overline{\mathbf{u}} < \overline{\mathbf{w}}$,
  - $\mathbb{L}_{\mathrm{Id}}(\alpha, \beta) \in P$,
  - $\mathbb{L}_{\mathrm{Id}}(\alpha - 1, \beta) \notin P$ or $\mathbb{L}_{\mathrm{Id}}(\alpha, \beta - 1) \notin P$.

Opération $F_{\mathrm{Id}}^{\alpha,\beta}$:

$$\begin{aligned}
\mathbf{o}' &= \mathbf{o} + \mathbf{u} \\
\mathbf{u}' &= \mathbf{u} + \alpha(\mathbf{u} - \mathbf{v}) + \beta(\mathbf{u} - \mathbf{w}) \\
\mathbf{v}' &= \mathbf{v} - \mathbf{u} \\
\mathbf{w}' &= \mathbf{w} - \mathbf{u}
\end{aligned}$$

$$M_{F_{\mathrm{Id}}^{\alpha,\beta}} := \begin{bmatrix} 1 + \alpha + \beta & -\alpha & -\beta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

Fully subtractive généralisé



Lemme

Soit $\mathfrak{S}$ système ne satisfaisant les préconditions d'aucune opération locale et σ telle que $\overline{\sigma(\mathbf{u})} < \overline{\sigma(\mathbf{v})}$ et $\overline{\sigma(\mathbf{u})} < \overline{\sigma(\mathbf{w})}$, alors il existe α, β tels que $\mathfrak{S}$ satisfait les préconditions de $F_\sigma^{\alpha,\beta}$.

Lemme

$F_\sigma^{\alpha,\beta}$ est valide sur un système satisfaisant ses préconditions.

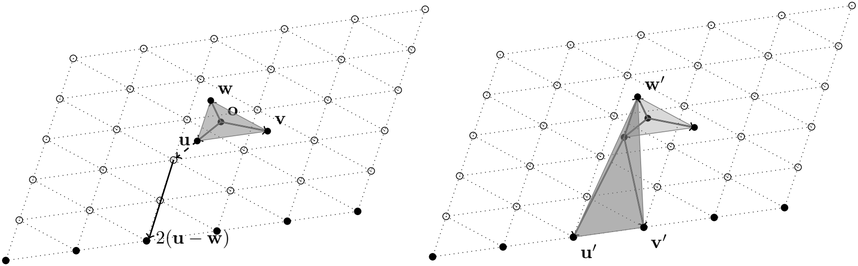
Brun généralisé

Opération définie pour un entier $\beta \geq 1$.

- Préconditions pour B_{Id}^{β} :
  - $\mathbf{o} + 2\mathbf{u}, \mathbf{o} + 2\mathbf{v}, \mathbf{o} + 2\mathbf{w} \notin P$,
  - $\overline{\sigma(\mathbf{u})} = \overline{\sigma(\mathbf{v})} < \overline{\sigma(\mathbf{w})}$.
  - $\mathbb{L}_{\sigma}(0, \beta) \in P$,
  - $\mathbb{L}_{\sigma}(0, \beta - 1) \in P$.
- Opération B_{Id}^{β}:

$$\begin{aligned} \mathbf{o}' &= \mathbf{o} + \mathbf{u} \\ \mathbf{u}' &= \mathbf{u} + \beta(\mathbf{u} - \mathbf{w}) \\ \mathbf{v}' &= \mathbf{v} + \beta(\mathbf{u} - \mathbf{w}) \\ \mathbf{w}' &= \mathbf{w} - \mathbf{u} \end{aligned} \qquad M_{B_{\mathrm{Id}}^{\beta}} := \begin{bmatrix} 1+\beta & 0 & -\beta \\ \beta & 1 & -\beta \\ -1 & 0 & 1 \end{bmatrix},$$

Brun généralisé



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Lemme

B_σ^β est valide sur un système satisfaisant ses préconditions.

The algorithm

Input: A predicate that answers the question : "*is* $\mathbf{x}$ *in* P" ?
Output: A system $\mathfrak{S} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{o})$
Select $\mathbf{o} \in P$ such that $\mathbf{o} + \{e_1, e_2, e_3\} \in P$;
$\mathfrak{S} \leftarrow (e_1, e_2, e_3, \mathbf{o})$;
stop $\leftarrow$ **False** ;
while ***not*** *stop* **do**
    if *there exist* σ *such that* $\mathfrak{S}$ *satisfies the preconditions of* T_σ **then**
        $\mathfrak{S} \leftarrow T_\sigma(\mathfrak{S})$;
    else if *there exist* σ *such that* $\mathfrak{S}$ *satisfies the preconditions of* B_σ **then**
        $\mathfrak{S} \leftarrow B_\sigma(\mathfrak{S})$;
    else if *there exists* σ *such that* $\mathfrak{S}$ *satisfies the preconditions of* F_σ **then**
        $\mathfrak{S} \leftarrow F_\sigma(\mathfrak{S})$;
    else if *there exists* σ *such that* $\overline{\sigma(\mathbf{u})} < \overline{\sigma(\mathbf{v})}$ *and* $\overline{\sigma(\mathbf{u})} < \overline{\sigma(\mathbf{w})}$ **then**
        Find (α, β) such that $\mathfrak{S}$ satisfied the preconditions of $F_\sigma^{\alpha,\beta}$;
        $\mathfrak{S} \leftarrow F_\sigma^{\alpha,\beta}(\mathfrak{S})$;
    else if *there exists* σ *such that* $\overline{\sigma(\mathbf{u})} = \overline{\sigma(\mathbf{v})} < \overline{\sigma(\mathbf{w})}$ **then**
        Find β such that $\mathfrak{S}$ satisfies the preconditions of B_σ^β ;
        $\mathfrak{S} \leftarrow B_\sigma^\beta(\mathfrak{S})$;
    else
        stop $\leftarrow$ **True**;
return $\mathfrak{S}$;

Validité de l'algotihme

- L'algorithme termine car $\overline{\mathbf{o}}$ est un entier borné qui augmente à chaque itération.
- Quand il termine :
  - $\overline{\mathbf{u}} = \overline{\mathbf{v}} = \overline{\mathbf{w}}$. Sinon, il exist σ tel que soit $\overline{\sigma(\mathbf{u})} < \min\left(\overline{\sigma(\mathbf{v})}, \overline{\sigma(\mathbf{w})}\right)$, ou $\overline{\sigma(\mathbf{u})} = \overline{\sigma(\mathbf{v})} < \overline{\sigma(\mathbf{w})}$.

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Lemme

Si un système valide est tel que $\overline{\mathbf{u}} = \overline{\mathbf{v}} = \overline{\mathbf{w}} = k$ alors $k = 1$ et $\hat{\mathbf{N}}(\mathfrak{S}) = \mathbf{N}$.

Preuve : soit $M = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}$ et $\mathbf{1} = e_1 + e_2 + e_3$.

- $k = 1$,
$$M\mathbf{N} = k\mathbf{1} \implies \mathbf{N} = kM^{-1}\mathbf{1}.$$
- $\hat{\mathbf{N}}(\mathfrak{S}) = \mathbf{N}$,
$$\langle \hat{\mathbf{N}}(\mathfrak{S}), \mathbf{u} \rangle = \langle \hat{\mathbf{N}}(\mathfrak{S}), \mathbf{v} \rangle = \langle \hat{\mathbf{N}}(\mathfrak{S}), \mathbf{w} \rangle = 1$$
Et donc, $M\mathbf{N} = M\hat{\mathbf{N}}(\mathfrak{S})$.

Examples

- $\mathbf{N} = (6, 8, 11)$,
- Opérations :
  - T_{uvw},
  - T_{uvw},
  - F_{uvw},
  - F_{vuw},
  - F_{vuw},
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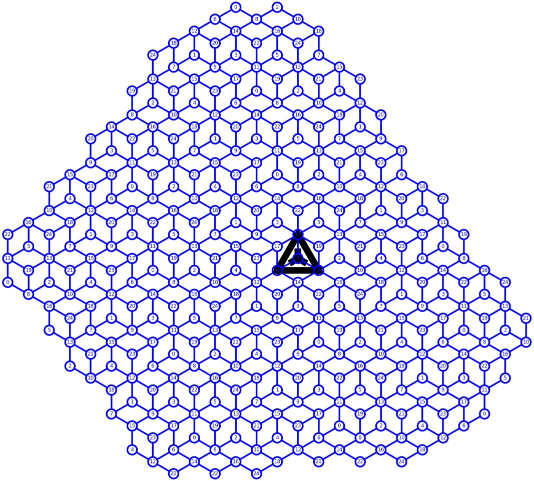

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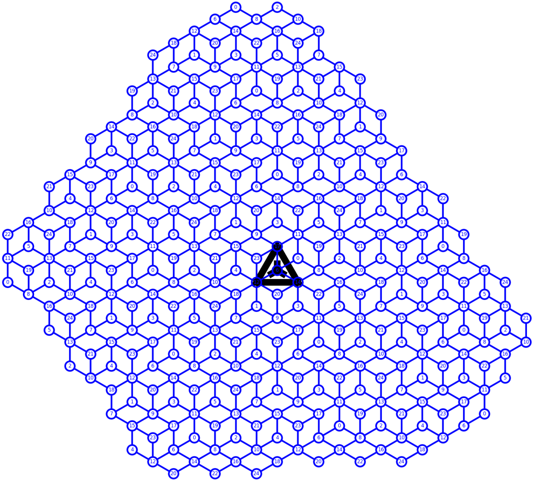

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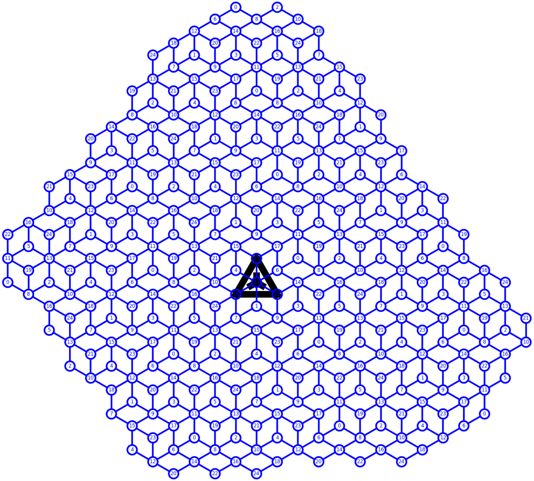

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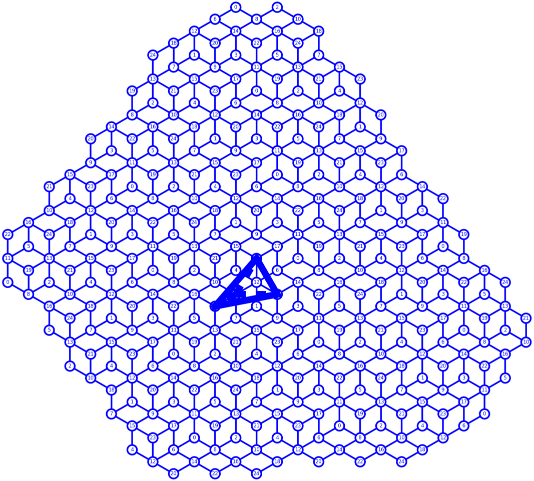

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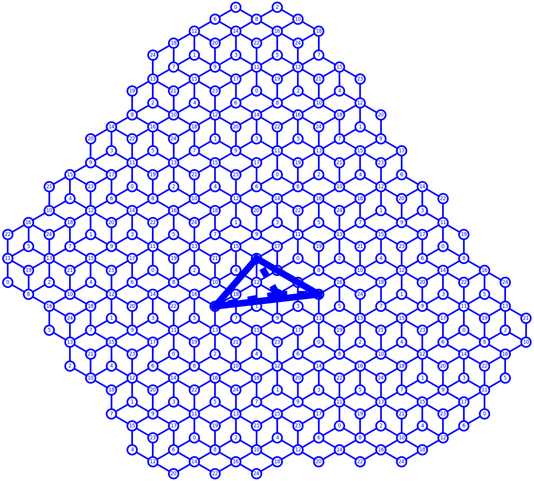

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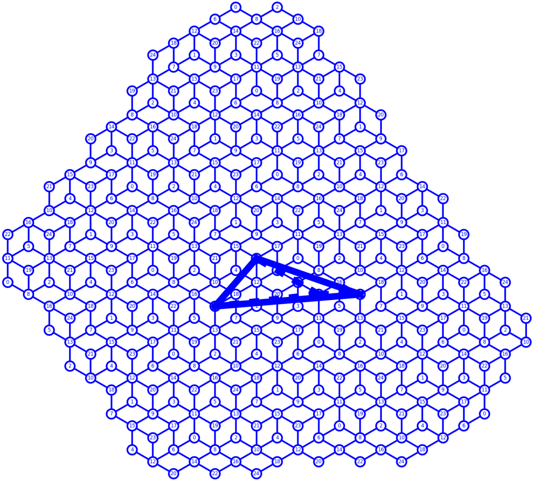

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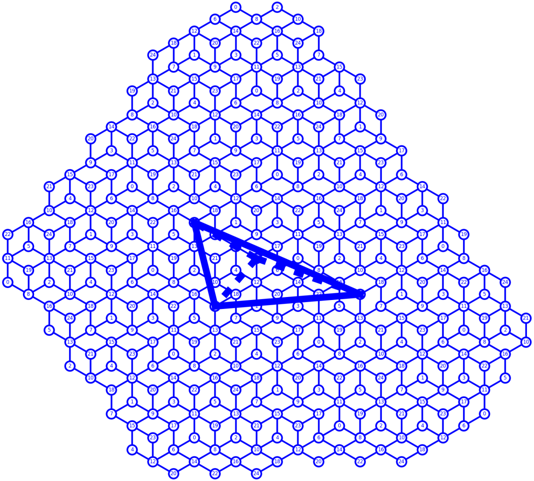

Examples

- $\mathbf{N} = (5, 7, 11)$,
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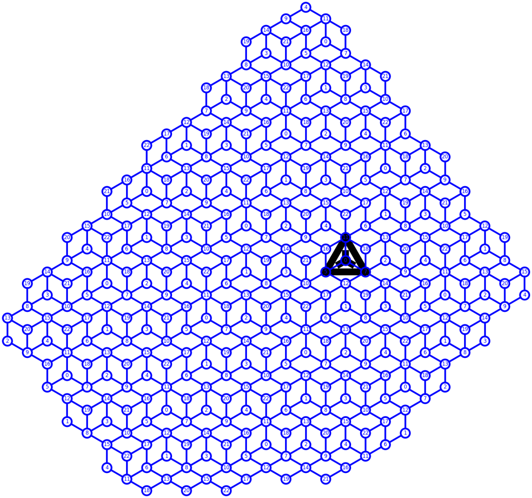

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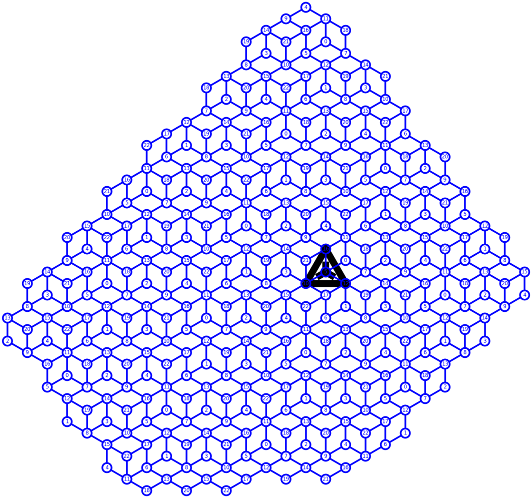

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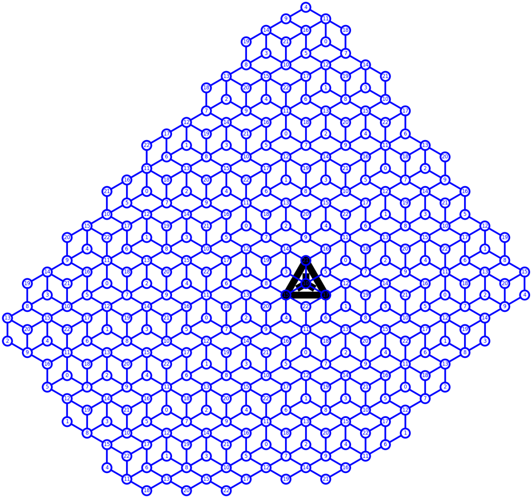

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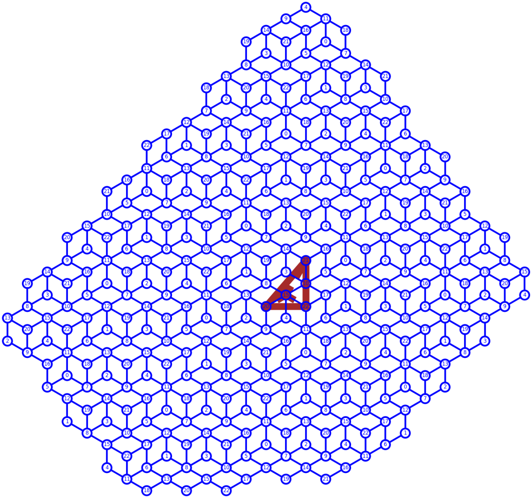

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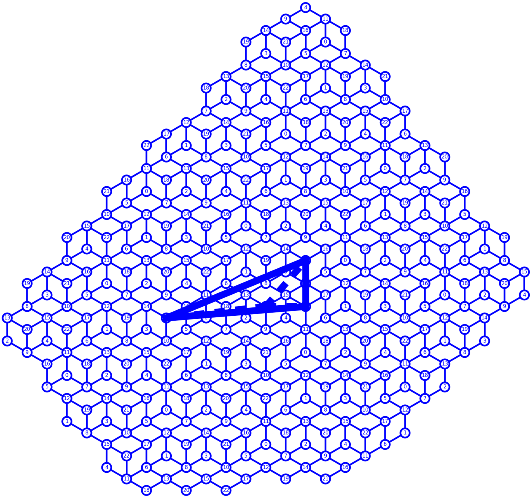

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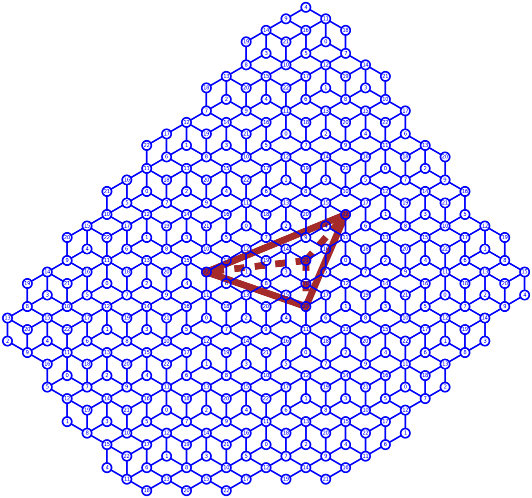

Version arithmétique

- On pose $\mathbf{N} = (a, b, c)$,
- À l'initialisation, on a $\mathfrak{S} = (e_1, e_2, e_3, \mathbf{o})$, ainsi, lorsqu'on compare $\overline{\mathbf{u}}, \overline{\mathbf{v}}, \overline{\mathbf{w}}$, on compare a, b, c.

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- Idée : réinterpréter l'algorithme de reconnaissance uniquement en fonction de $\overline{\mathbf{u}}, \overline{\mathbf{v}}, \overline{\mathbf{w}}$ et $\Omega_0 = \omega - \overline{\mathbf{o}}$.
- Pour simplifier, on considère la version triée :

$$\overline{\mathbf{u}} \leq \overline{\mathbf{v}} \leq \overline{\mathbf{w}} < \Omega_0.$$

Version arithmétique

- Si $a = b = c$ alors STOP.
- Sinon si $a + c < \Omega_0$ alors Translation : $(a, b, c, \Omega_0) \leftarrow (a, b, c, \Omega_0 - a)$
- Sinon si $a + b < \Omega_0$ alors Brun/Selmer : $(a, b, c, \Omega_0) \leftarrow \mathbf{sort}(a, b, c - a, \Omega_0 - a)$
- Sinon si $2a < \Omega_0$ alors FS : $(a, b, c, \Omega_0) \leftarrow \mathbf{sort}(a, b - a, c - a, \Omega_0 - a)$
- Sinon
  - On pose :
    - $\Gamma(x) = \dfrac{\Omega_0}{x+1} + c\dfrac{x-1}{x+1}$,
    - β tel que : $\Gamma(\beta) \leq a < \Gamma(\beta + 1)$.
  - Si $a \neq b$ alors FS généralisé :

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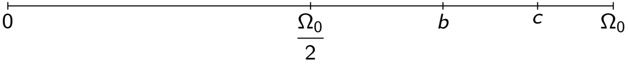

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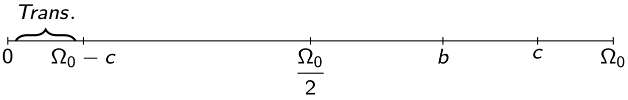

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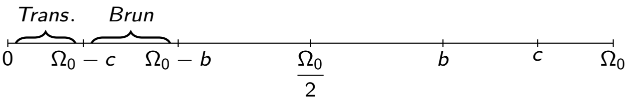

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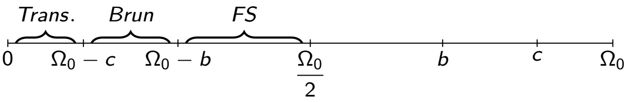

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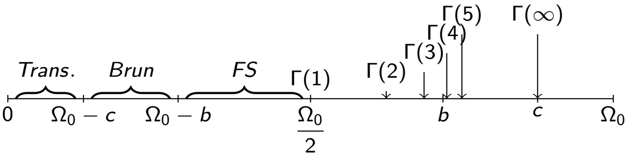

Examples

- $\mathbf{N} = (5, 16, 15)$,
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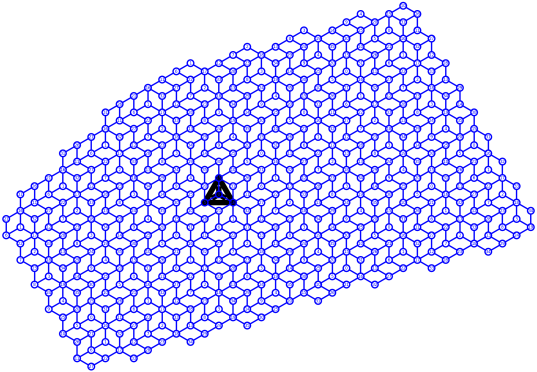

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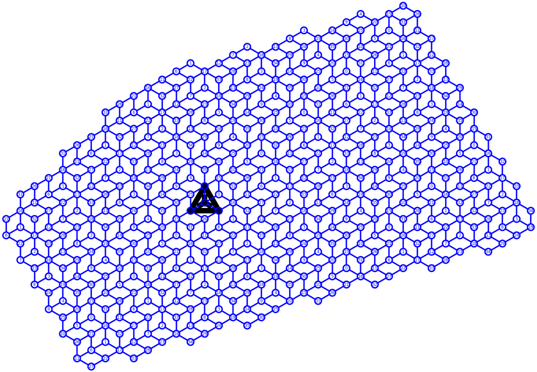
  - F^2_{uvw},
  - F^4_{vwu},



Examples

- $\mathbf{N} = (5, 16, 15)$,
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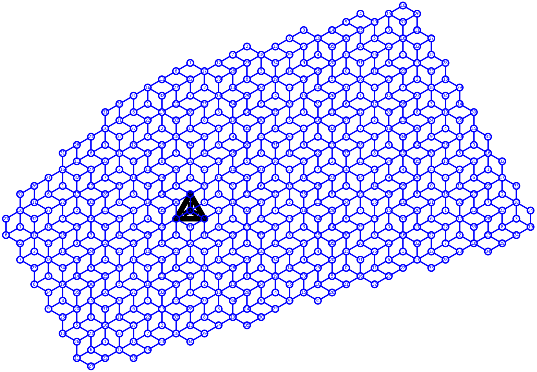

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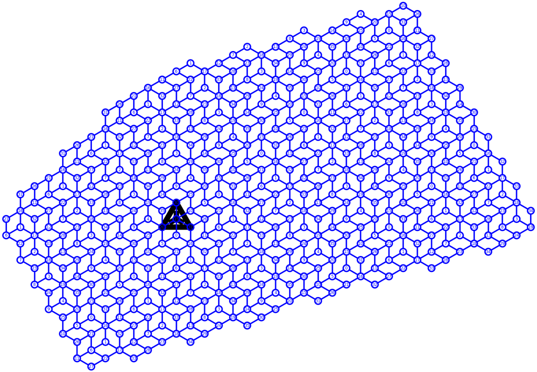

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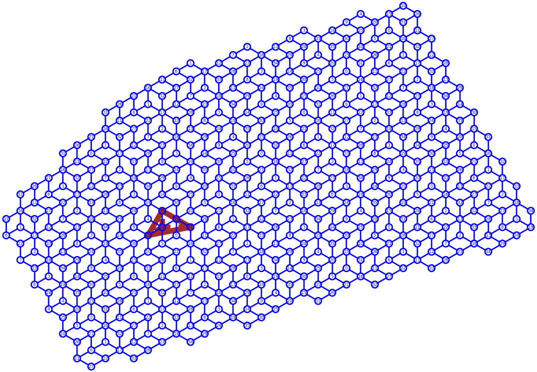

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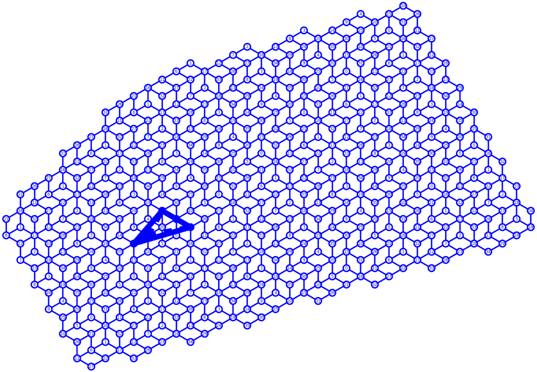

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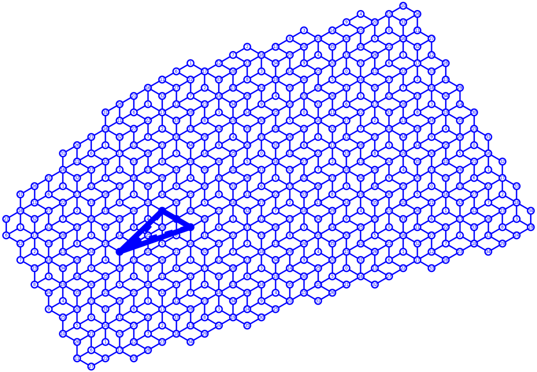

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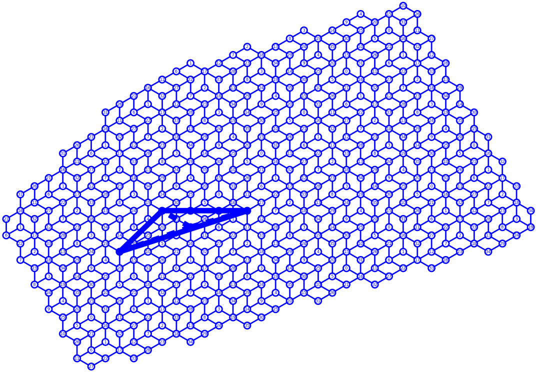

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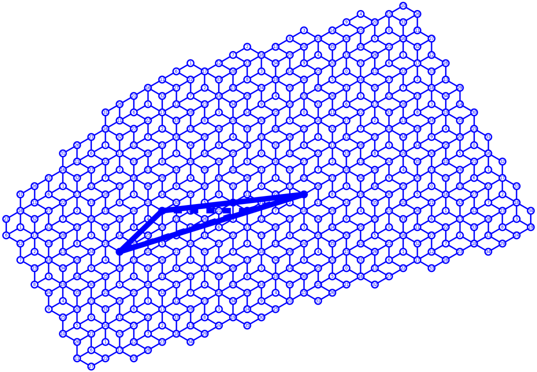

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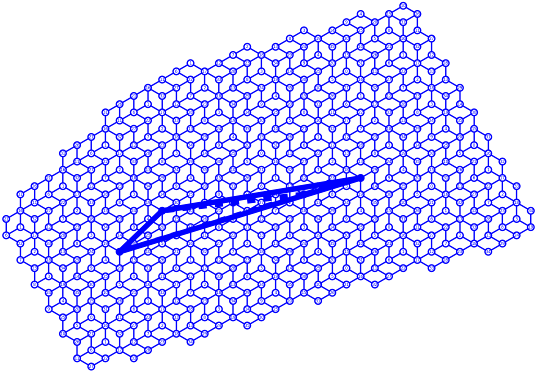

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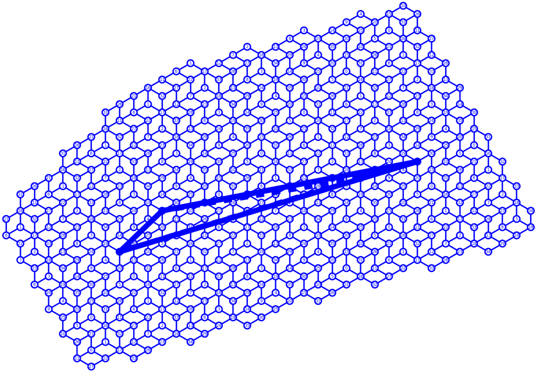

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  - B_{uvw},
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  - D_{wvu},
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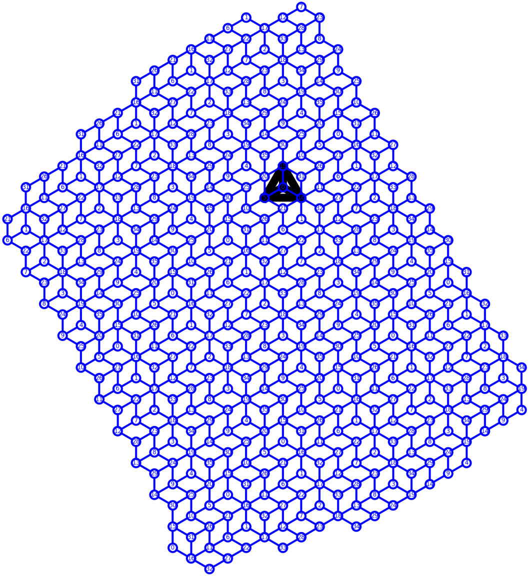

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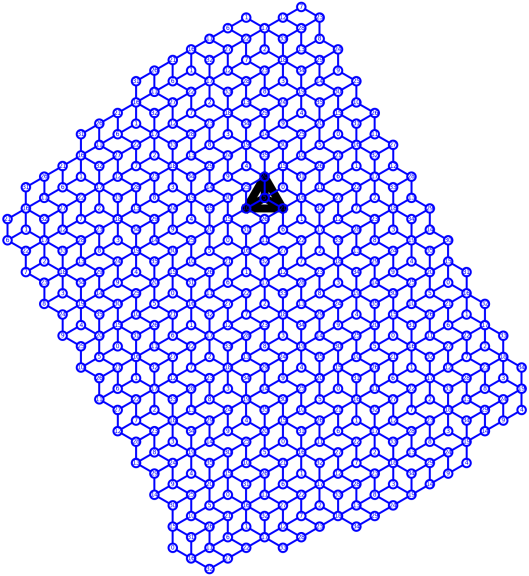

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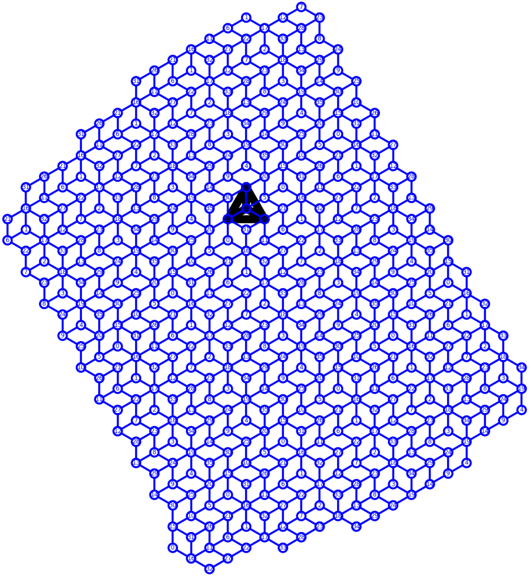

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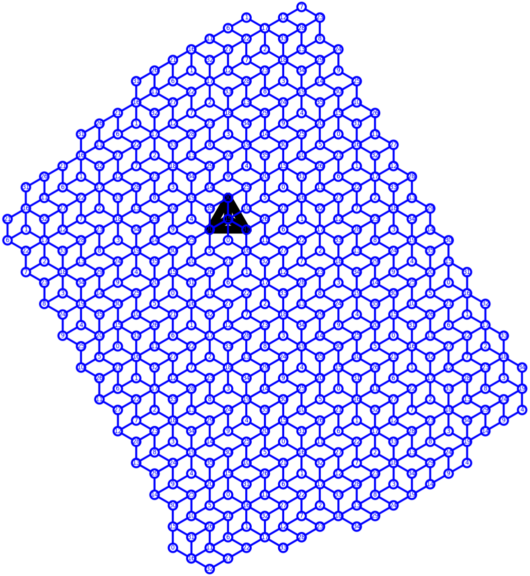

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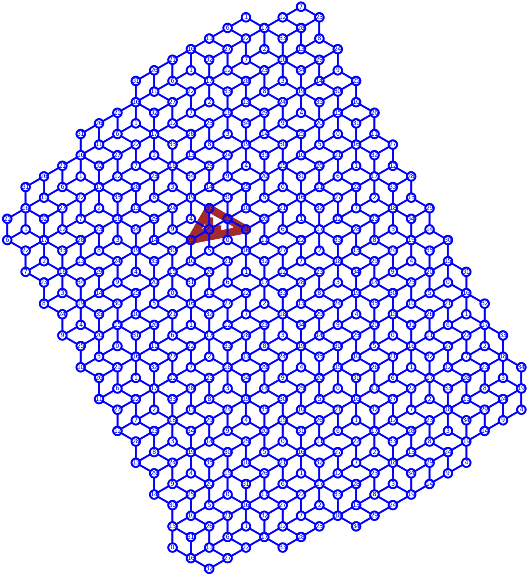

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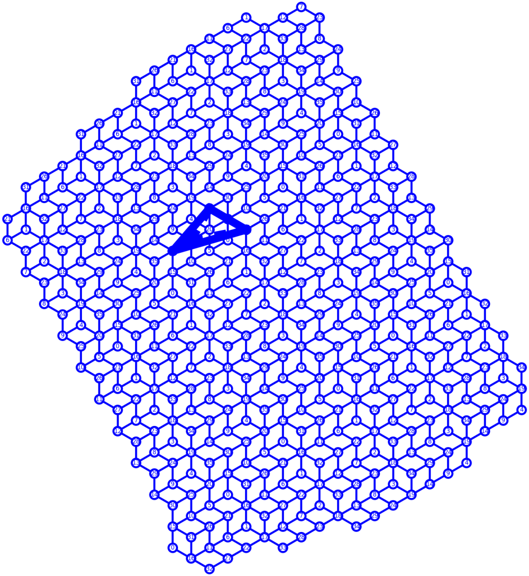

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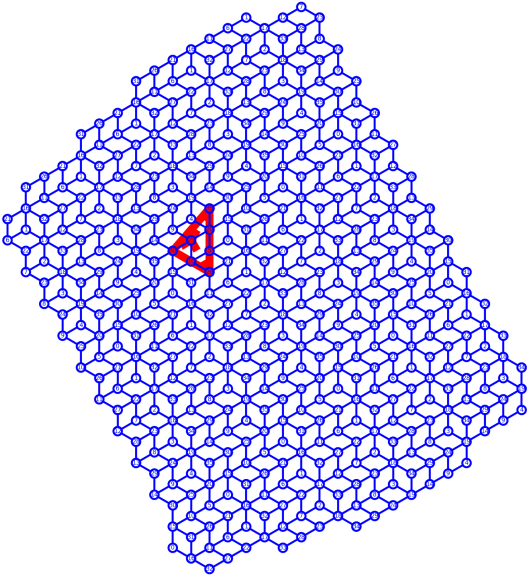

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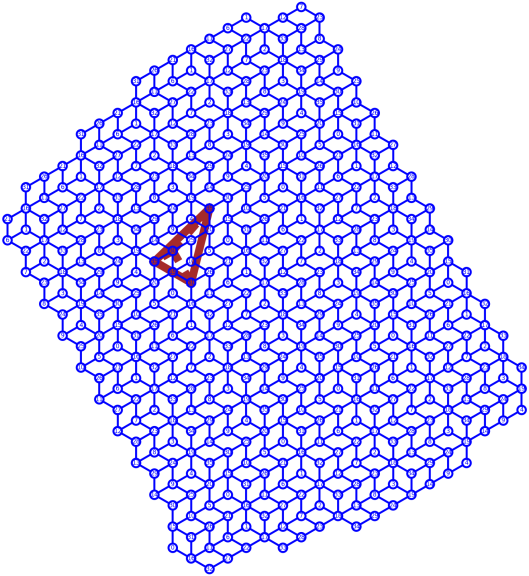

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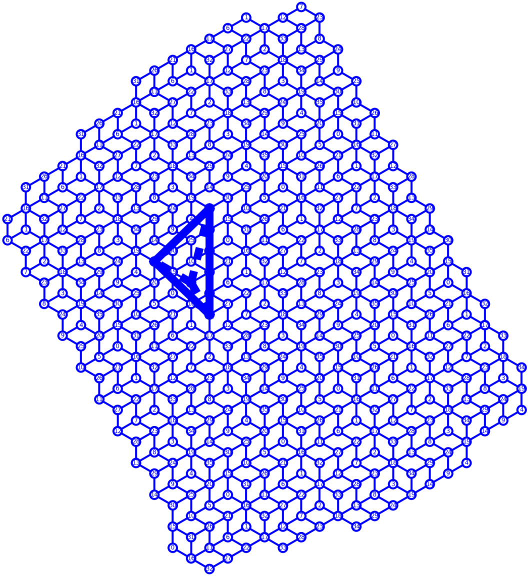

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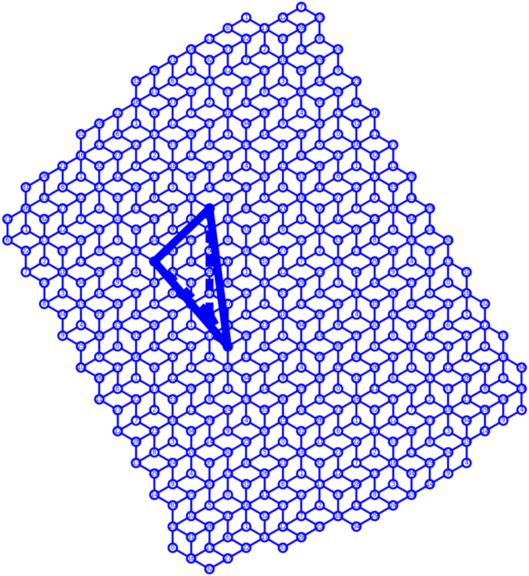

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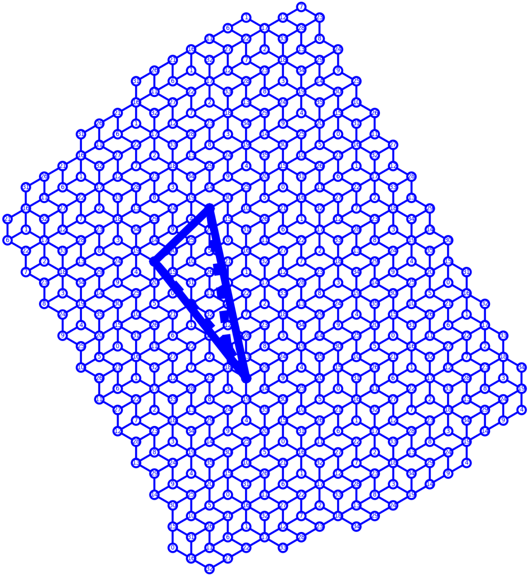

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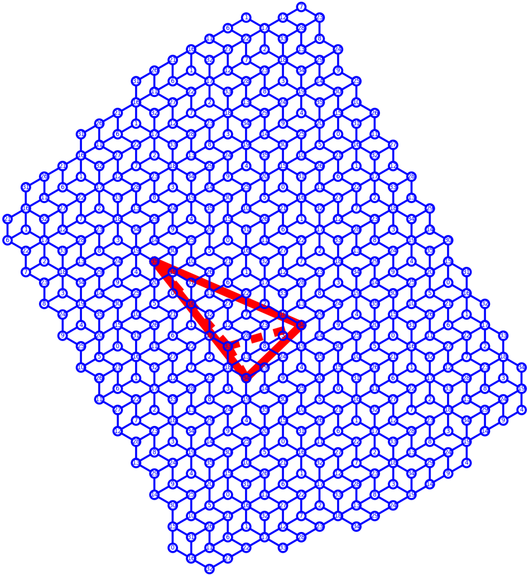

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