Structutre des plans discrets minces

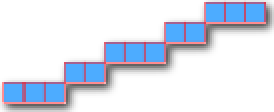
X. Provençal,

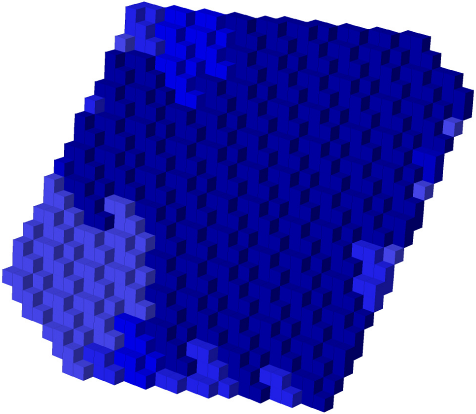
collaboration avec V. Berthé, É. Domenjoud, D. Jamet, T. Jolivet

LAMA
Laboratoire de Mathématiques
Université de Savoie

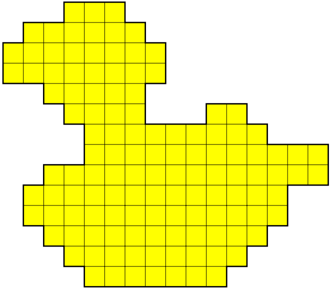
SIGILLUM UNIVERSITATIS SABAUDIENSIS

23 juillet 2020





Minimum Length Polygon



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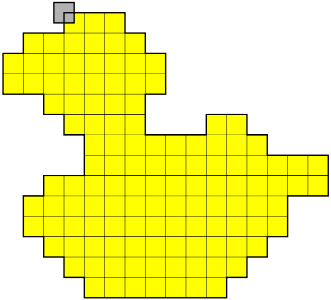

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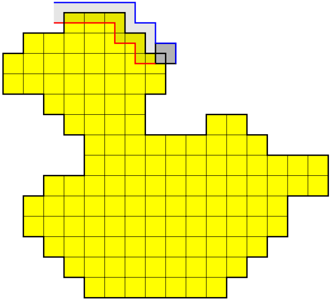

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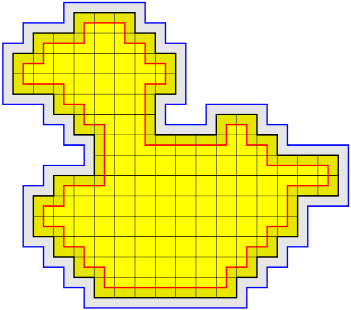
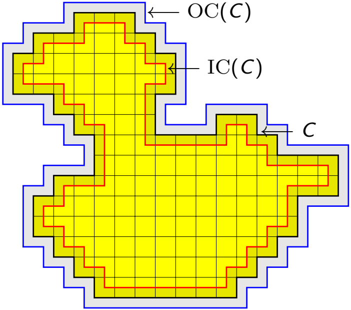


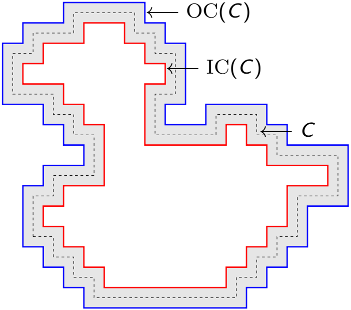


Définition (Montarani (1970) et Sklansky Chazin, Hansen (1972))

The *minimum length polygon* of C is a subset $P \in \mathbb{R}^2$ such that,

$$P = \underset{A \in \mathcal{A},\, \mathrm{IC}(C) \subseteq A,\ \partial A \subset \mathrm{OC}(C) \setminus \mathrm{IC}(C)^\circ}{\arg\min} \mathrm{Per}(A)$$

where $\mathcal{A}$ is the family of simply connected compact sets of $\mathbb{R}^2$.



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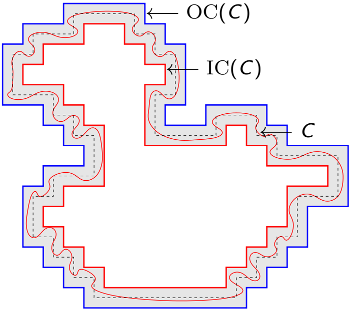

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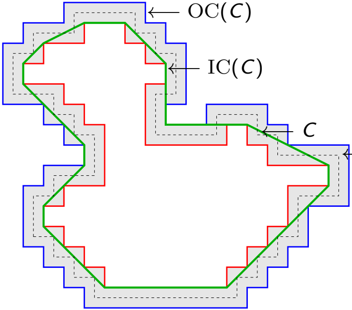

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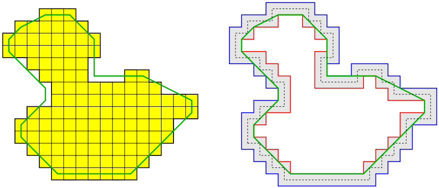
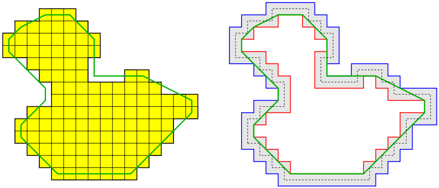


The MLP is a polygonal line whose vertices are centers of pixels along the inner or the outer contour, also :

- unique ;
- a good length estimator[1] ;
- a good tangent estimator ;
- characteristic of the shape's convexity ;
- reversible[2].

[1] Proved to be convergent on convex shapes.
[2] If aligned vertices are considered.



MLP is computable in time linear with respect of the length of C.

- J.-O. Lachaud, X. Provençal, *Two linear-time algorithms for computing the minimum length polygon of a digital contour*, Discrete Applied Mathematics (DAM), 2011.
- Tristan Roussillon, Isabelle Sivignon. *Faithful polygonal representation of the convex and concave parts of a digital curve*. Pattern Recognition, volume 44, issues 10-11, p. 2693-2700, 2011.

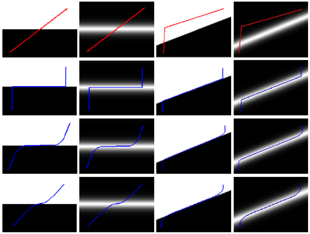
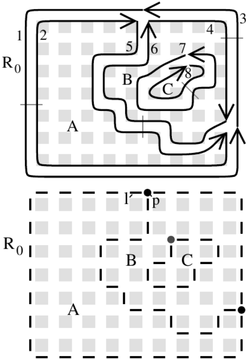
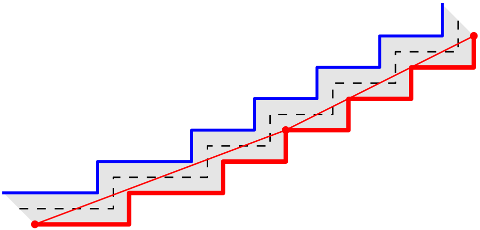


Fig. 4. Example of the minimization process using the Greedy1 algorithm. The gradient is computed with the Canny-Deriche method with scale coefficient 0.2. The input image represents a half-plane. (First row) Initialisation of the DDM. (Second row) Results of the minimisation process, the α coefficient used is equal to 0. (Third row) Results with $\alpha = 200$. (Fifth row) Results with $\alpha = 300$.



- F. de Vieilleville and J.-O. Lachaud, *Digital Deformable Model Simulating Active Contours*, in proc. DGCI2009, LNCS 5810, p.203-216, 2009.
- G. Damiand, A. Dupas and J.-O. Lachaud, *Combining Topological Maps, Multi-Label Simple Points, and Minimum-Length Polygons for Efficient Digital Partition Model*, in proc. IWCIA2011, LNCS 6636, p. 208-221, 2011.

$(8,3) \otimes (2,1)^3 = 2.$



$[(8,3),(2,1)^3]$

- J.-O. Lachaud, X. Provençal, *Dynamic Minimum Length MLP*, in proc. IWCIA2011, LNCS 6636, p. 208-221, 2011.

Split and merge case : $x \otimes y > 1$

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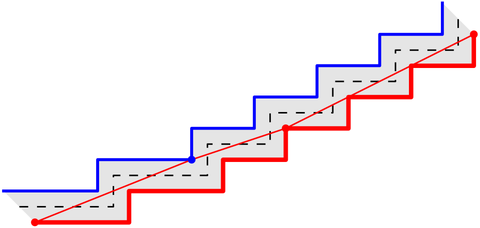

$[(8,3),(2,1)^3] \equiv [\widetilde{(2,5)},\widetilde{(3,1)},(2,1)^3]$

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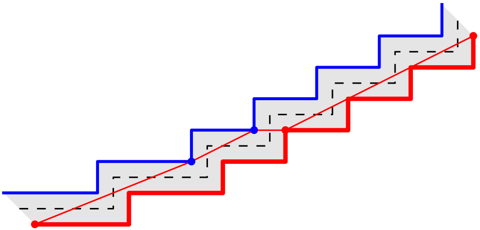

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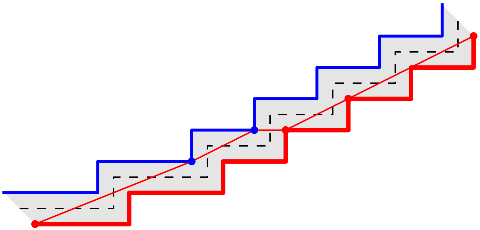

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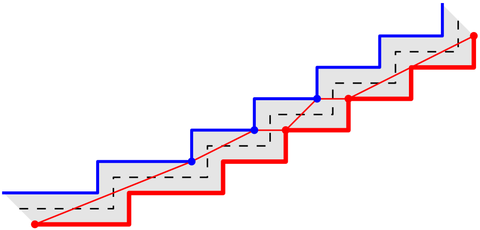

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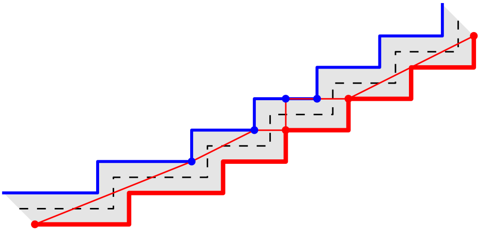

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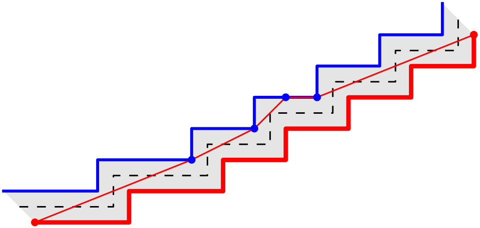

$[(8,3),(2,1)^3] \quad \equiv [\widetilde{(2,5)},(1,2),(1,1)(0,1),\widetilde{(5,2)}]$

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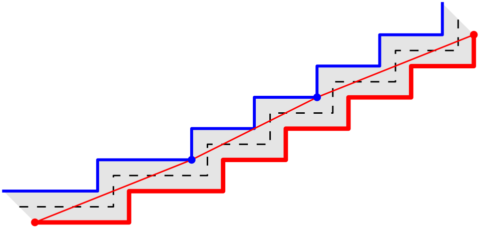

$[(8,3),(2,1)^3] \quad \equiv [\widetilde{(2,5)},(1,2)^2,\widetilde{(5,2)}]$

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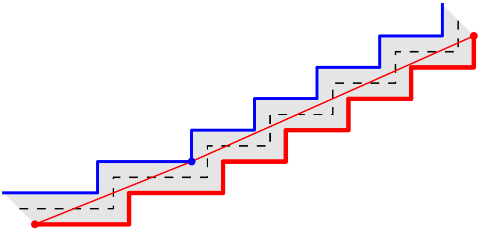

$[(8, 3), (2, 1)^3] \quad \equiv [\widetilde{(2, 5)}, \widetilde{(9, 4)}]$

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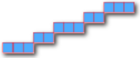

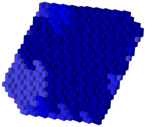

Définition (J.-P. Reveillès, 1991)

L'hyperplan discret $P(v, \mu, \omega)$ de *vecteur normal* $v \in \mathbb{R}^n$, de *décalage* $\mu \in \mathbb{R}$ et d'*épaisseur* $\omega \in R_+$ est le sous-ensemble de $\mathbb{Z}^n$ défini par :

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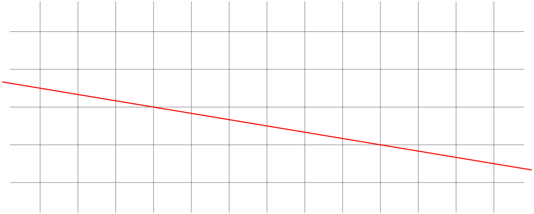

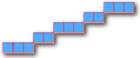

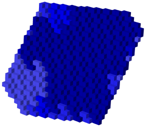

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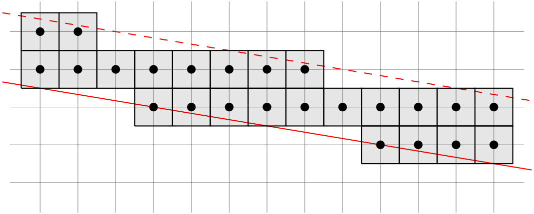

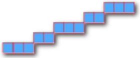

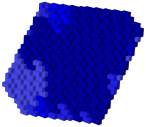

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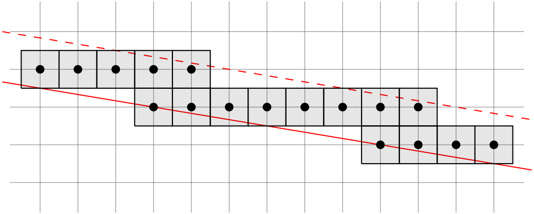

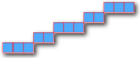

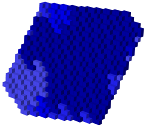

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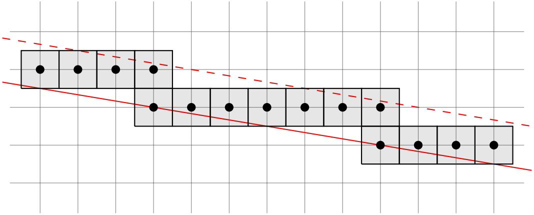

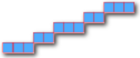

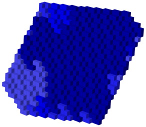

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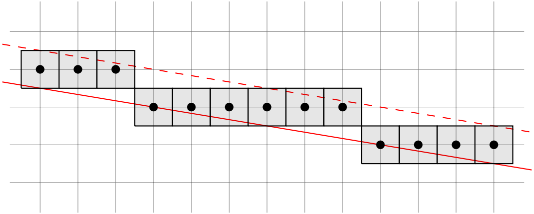

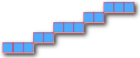

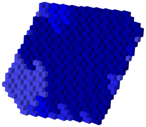

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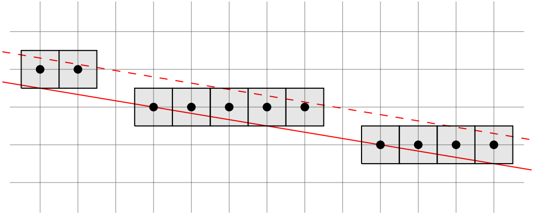
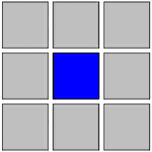


Connexité discrète

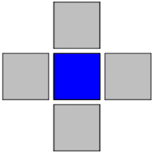
Définition

Deux points x, y de $\mathbb{Z}^d$ sont *k-connexes* si

$$\|x-y\|_\infty \leq 1 \quad \text{et} \quad \|x-y\|_1 \leq d-k.$$



0-voisins



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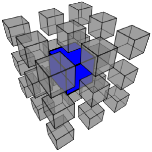

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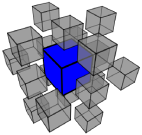

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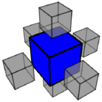
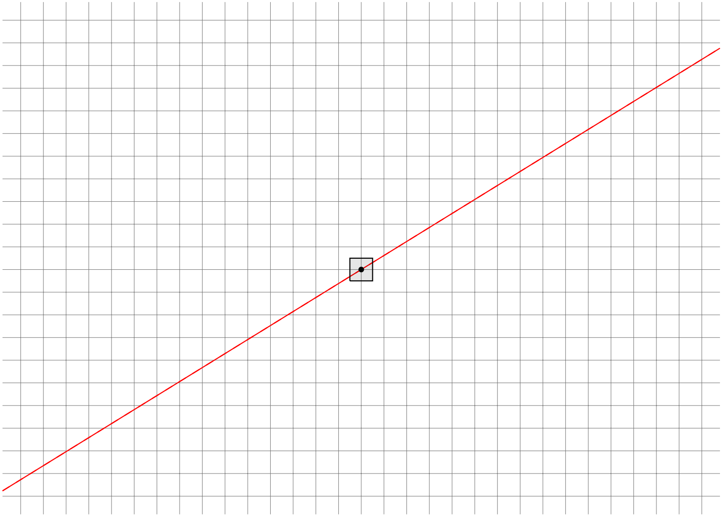


2-voisins

Droite de Fibonacci



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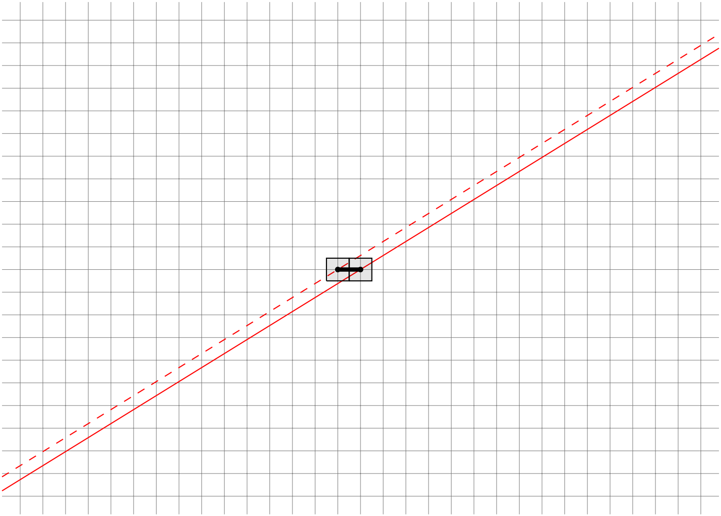
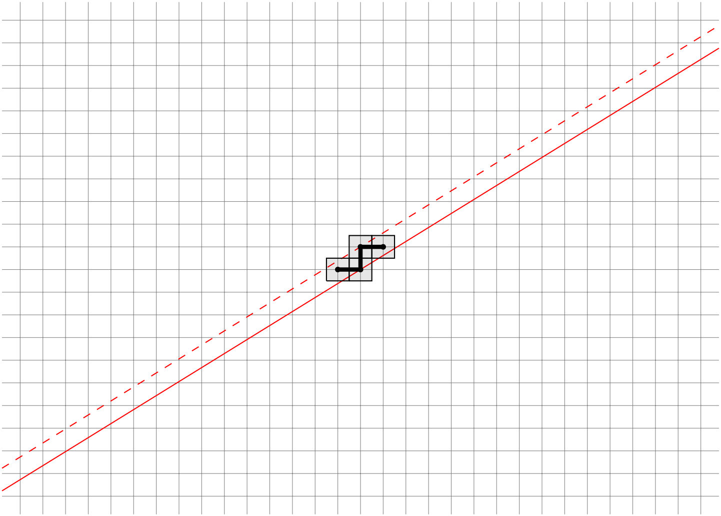
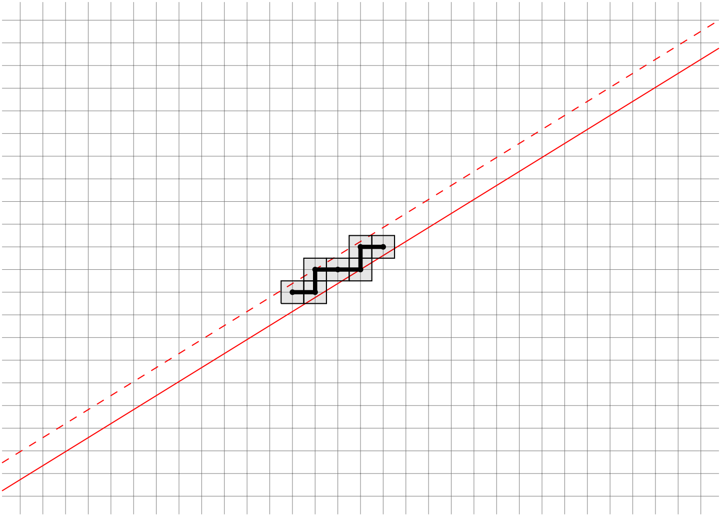


Droite de Fibonacci





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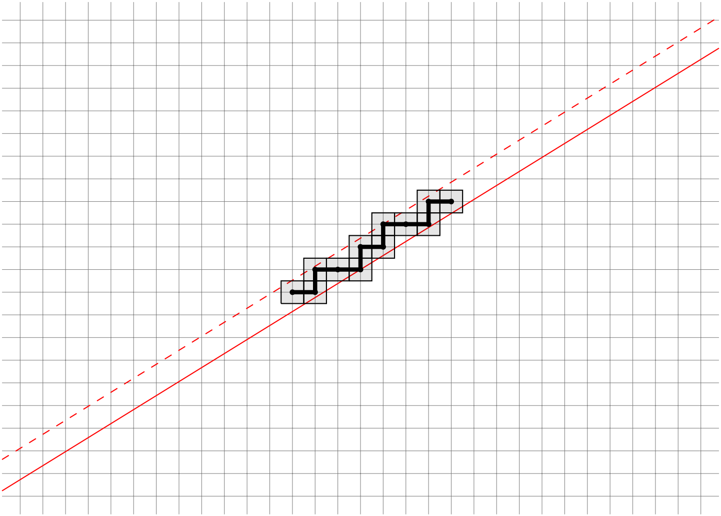

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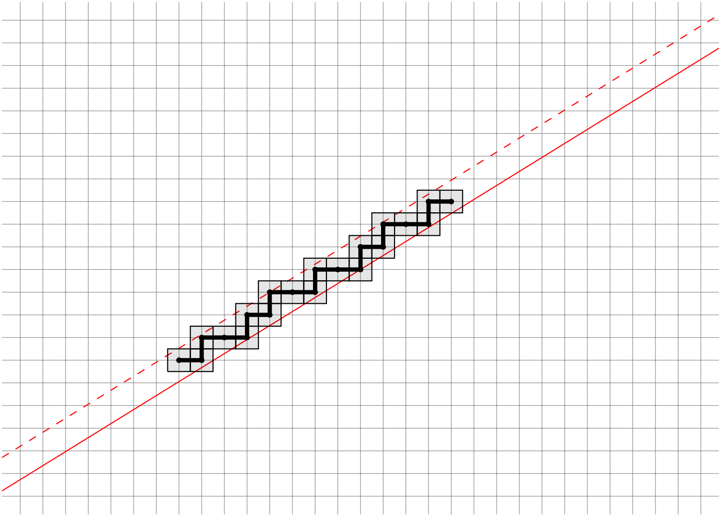

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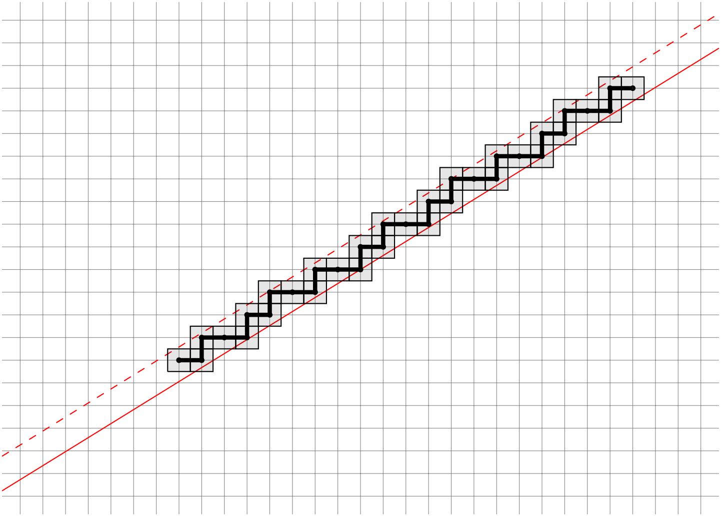

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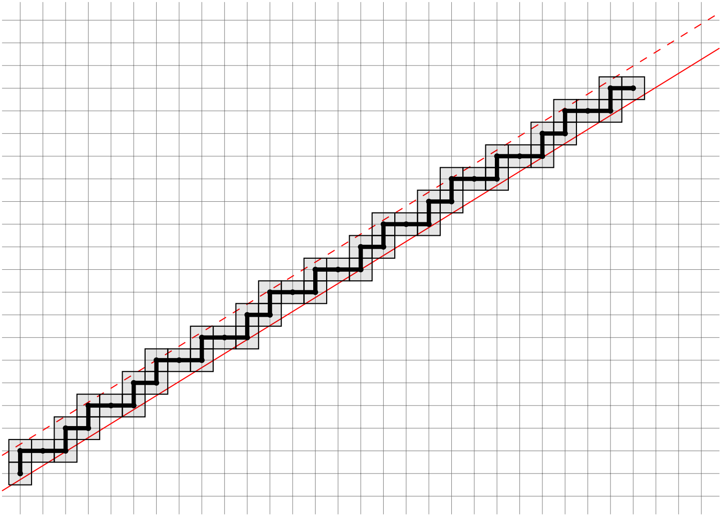

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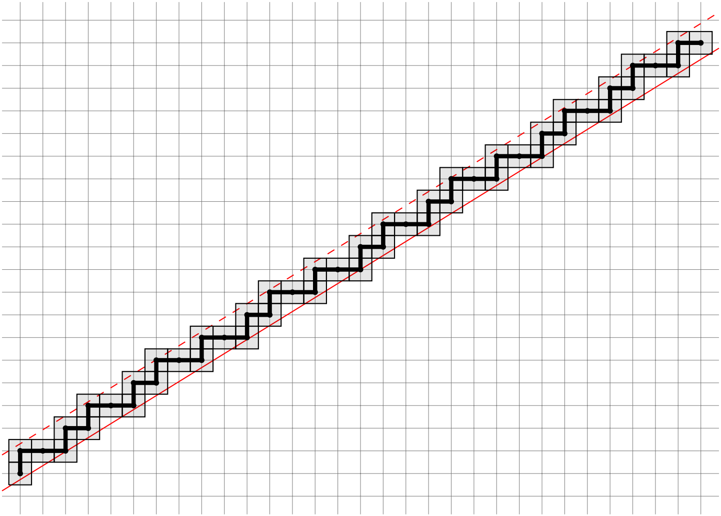
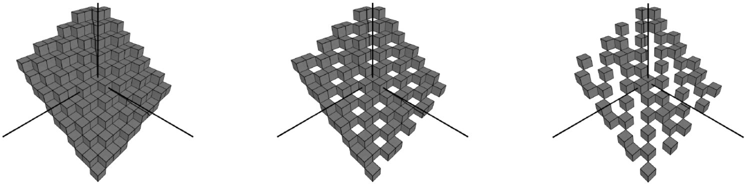


Question

On considère un vecteur $v \in \mathbb{R}^d$, un réel $\mu \in \mathbb{R}$ et un entier $\kappa \in \{0, 1, \cdots, d-1\}$.

Déterminer $\Omega_\kappa(v, \mu) = \inf \{\omega \in \mathbb{R} \mid P(v, \mu, \omega) \text{ est } \kappa\text{-connexe}\}$



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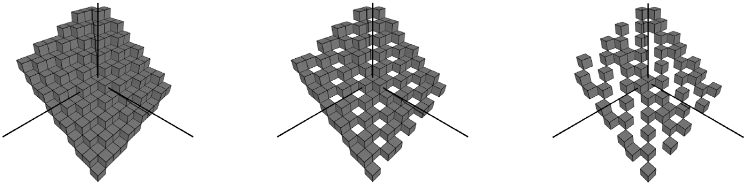

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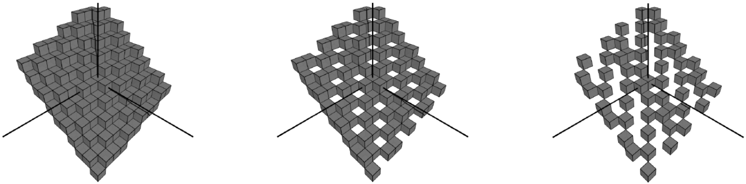

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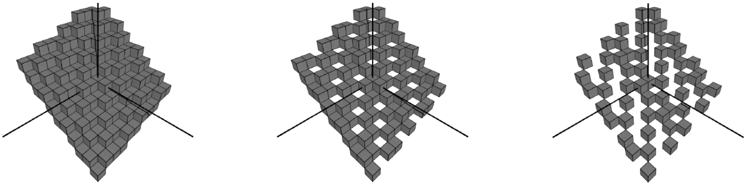

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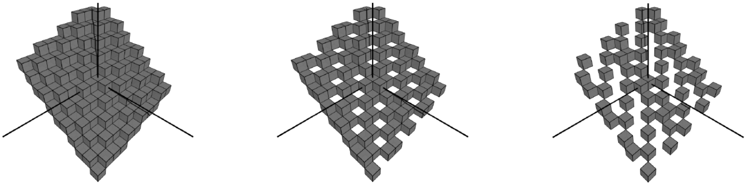

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Théorème (Domenjoud, Jamet, Toutant, 2009)

Soit $v = (v_1, v_2, v_3) \in \mathbb{R}_+^3$, avec $v_1 \leq v_2 \leq v_3$, $\mu \in \mathbb{R}_+$ alors

$$\Omega_2\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \mu\right) = \Omega_2\left(\begin{pmatrix} v_1 \\ v_2 - v_1 \\ v_3 - v_1 \end{pmatrix}, \mu\right) + v_1 .$$

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Démonstration.

On se rappelle que $\langle x, v\rangle = \langle {}^tM^{-1}x, Mv\rangle$ et on pose

$$M = \left[\begin{array}{rcc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right], \quad P_1 := P(v, \mu, \omega), \quad P_2 := P(Mv, \mu, \omega - v_1).$$

On montre que

$$P_1 \text{ est 2-connexe} \iff P_2 \text{ est 2-connexe}.$$

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Étant donné un vecteur $v \in \mathbb{R}^3_+$, on pose $v^{(0)} = v$, $\omega^{(0)} = 0$ et pour $n \geq 1$,

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Calcul de l'épaisseur minimale

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  - Sinon $\Omega_2(v,\mu) = \frac{v_1+v_2+v_3}{2}$.

$$v^{(n+1)} = M^{(n)} v^{(n)}.$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$M^{(n)} = \begin{cases} M_1 & \text{si } v_1^{(n)} < v_2^{(n)} - v_1^{(n)} \\ M_2 & \text{si } v_2^{(n)} - v_1^{(n)} < v_1^{(n)} \\ M_3 & \text{si } v_3^{(n)} - v_1^{(n)} < v_1^{(n)} \end{cases}$$

Plan de Tribonacci

Soit α la racine réelle de $x^3 + x^2 + x - 1$.

On pose $v = (\alpha, \alpha + \alpha^2, 1)$. On a alors que

$$M_3 v = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ \alpha + \alpha^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^2 \\ 1 - \alpha \\ \alpha \end{pmatrix} = \alpha v$$

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$$\Omega_2(v, 0) = \alpha + \alpha^2 + \alpha^3 + \cdots = \frac{\alpha}{1 - \alpha}.$$

On pose $P_0 = \{(0,0,0)\}$
et pour tout $n \geq 1$,

$$\mathcal{P}_n = \cup_{i=0}^{n} P_i,$$

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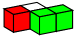

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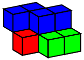

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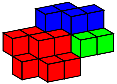

Construction itérative du plan de Tribonacci

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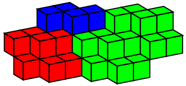

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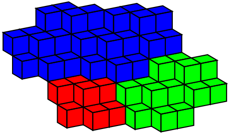

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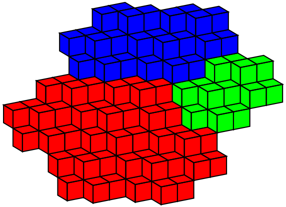

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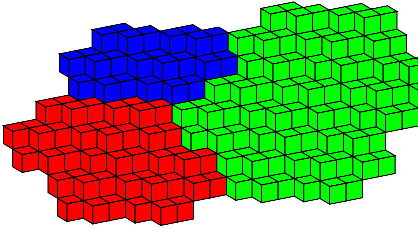

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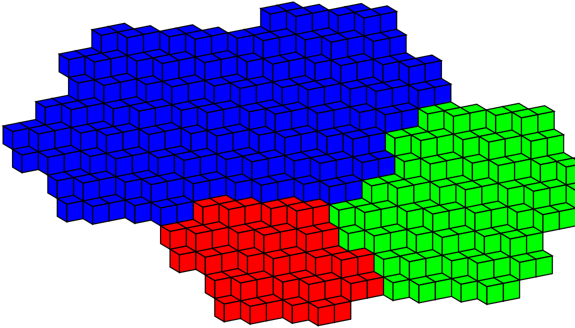

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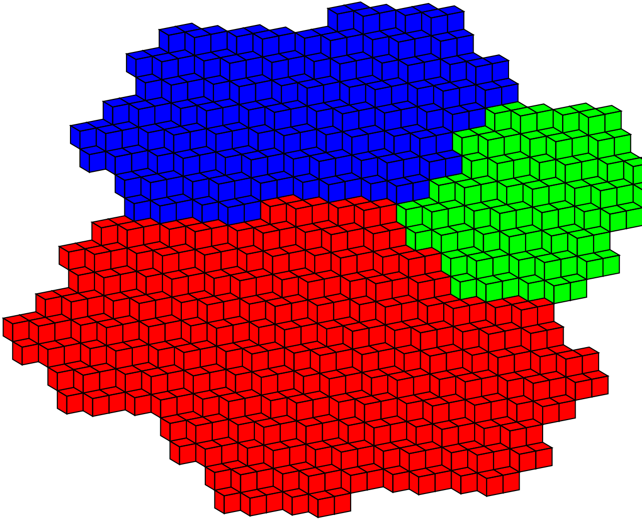

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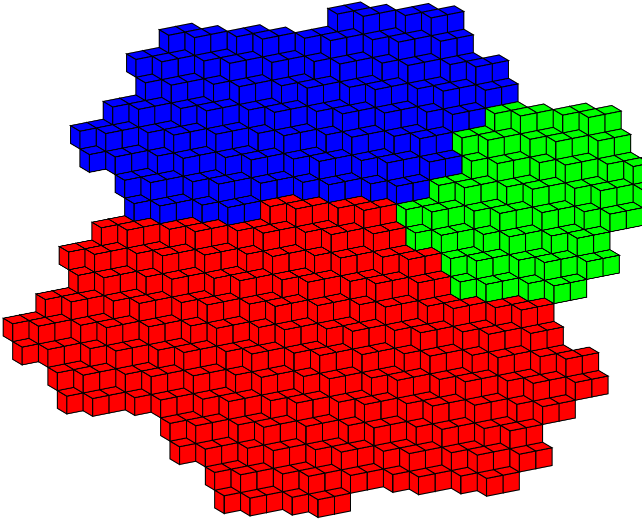

- $x \in P_n \implies \langle x, v\rangle = \alpha^n + \sum_{i=0}^{n-1} \delta_i \alpha^i$, où chaque $\delta_i \in \{0,1\}$.
- $P_n \subset P_{n+3}$

Théorème

$$\lim_{n\to\infty} \mathcal{P}_n = P\left(\left(\alpha, \alpha^2+\alpha, 1\right), 0, \Omega_2(v,0)\right).$$

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Preuve

On a $\Omega_2(v,0) = \frac{\alpha}{1-\alpha}$.

- $\subset$
  $x \in \mathcal{P}_n \implies 0 \leq \langle x, v\rangle < 1+\alpha+\alpha^2+\cdots$.

Théorème

$$\lim_{n\to\infty} \mathcal{P}_n = P\left(\left(\alpha, \alpha^2+\alpha, 1\right), 0, \Omega_2(v,0)\right).$$

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  $x \in \mathcal{P}_n \implies 0 \leq \langle x, v\rangle < 1 + \alpha + \alpha^2 + \cdots$.
- $\supset$
  Dans le contexte des systèmes de numération, (Frougny, Solomyak, 1992),

$$\forall x \in \mathbb{Z}^3, \exists (\delta_i)_{-n\leq i\leq n}, \delta_i \in \{0,1\} \text{ tel que } \sum_{i=-n}^{n} \delta_i \alpha^i = \langle x, (\alpha, \alpha+\alpha^2, 1)\rangle.$$

Théorème

$$\lim_{n\to\infty} \mathcal{P}_n = P\left(\left(\alpha, \alpha^2+\alpha, 1\right), 0, \Omega_2(v,0)\right).$$

Preuve

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Propriétés

- Le plan construit est 2-connexe.

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- Pour tout $n \geq 0$, $\mathcal{P}_n$ est un arbre.

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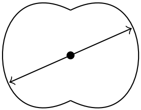
Propriétés

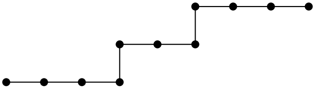
- Le plan construit est 2-connexe.
- Pour tout $n \geq 0$, $\mathcal{P}_n$ est un arbre.
- Pour tout $n \geq 0$, $\mathcal{P}_n$ est centrosymétrique.

Définition

Un ensemble $S \subset \mathbb{R}^d$ est dit *centrosymétrique* de centre c si

$$c + x \in S \implies c - x \in S.$$





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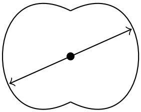

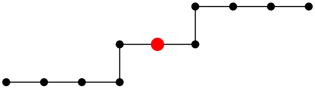

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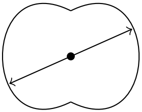

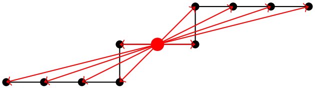

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Construction par translation et union

Étant donné un vecteur $v \in \mathbb{R}^3_+$ avec $v_1 \leq v_2 \leq v_3$, on a

$$v^{(n+1)} = M^{(n)} v^{(n)},$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

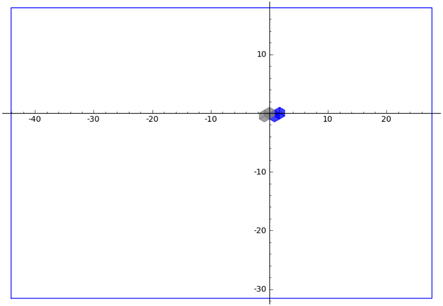
On pose $\mathcal{M}_0 = Id$, pour tout $n \geq 1$,

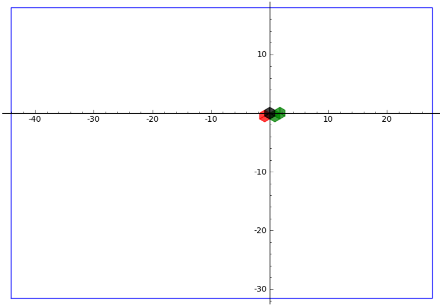
$$\mathcal{M}_n = M^{(n-1)} \mathcal{M}_{(n-1)} \text{ et } \vec{t_n} = (1,0,0)\mathcal{M}^n.$$

On pose $P_0 = \{(0,0,0)\}$ et pour tout $n \geq 1$,

$$\begin{aligned} \mathcal{P}_n &= \cup_{i=0}^{n} P_i, \\ P_n &= \mathcal{P}_{n-1} + \vec{t_n}, \end{aligned}$$

On pose $v = (67, 107, 129)$, on obtiens la emphsuite directrive $[3, 3, 3, 1, 1, 2, 2, 3, 3, 3]$.





Gris : $\mathcal{P}_{n-1} \setminus (\mathcal{P}_{n-1} + \vec{t_n})$
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Exemple

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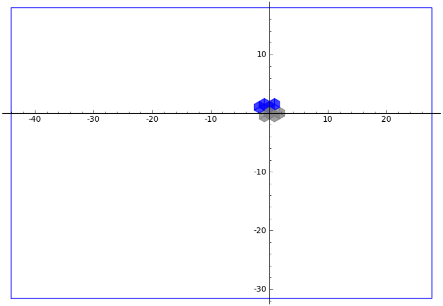

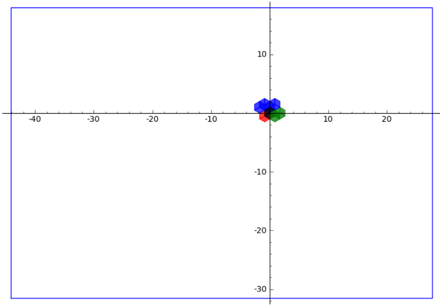

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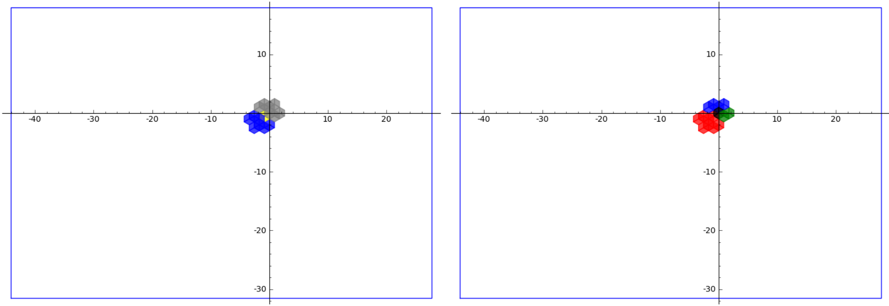

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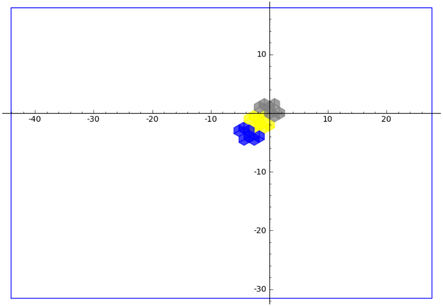

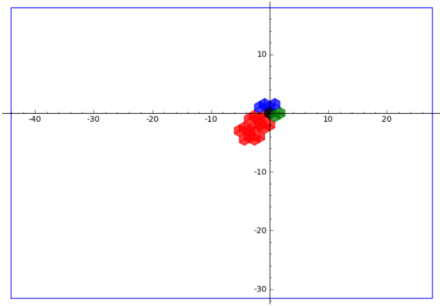

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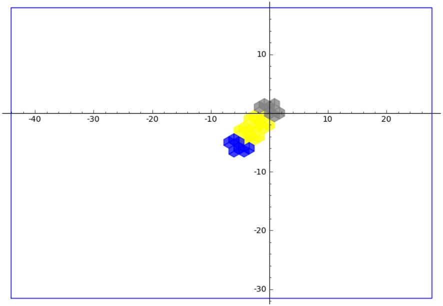

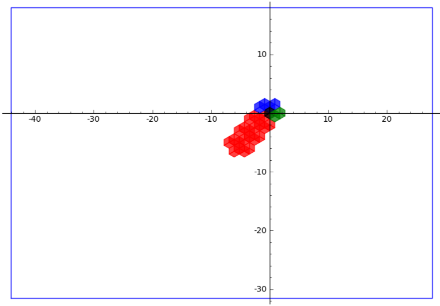

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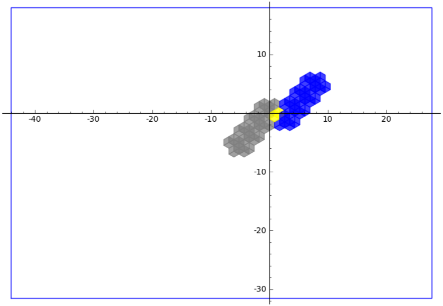

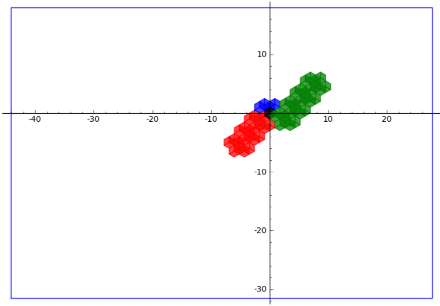

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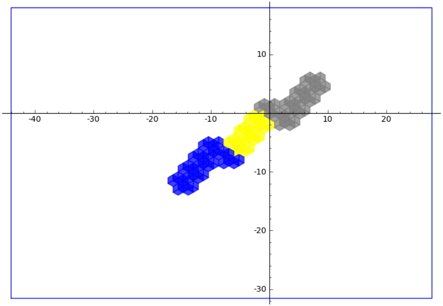

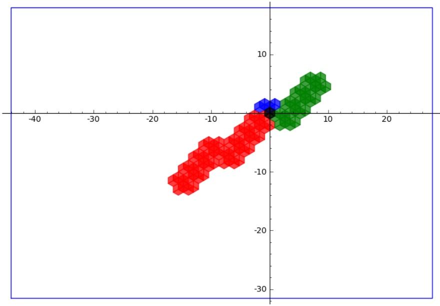

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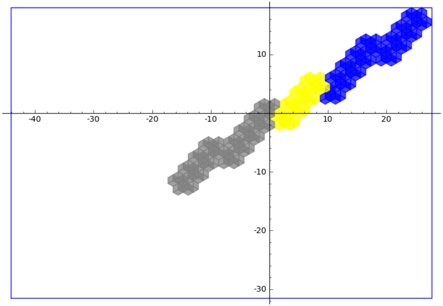

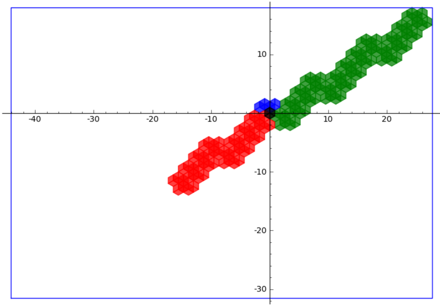

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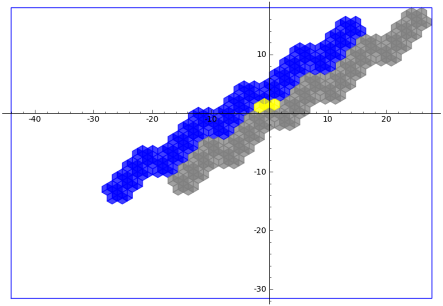

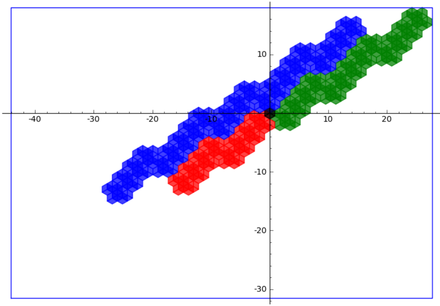

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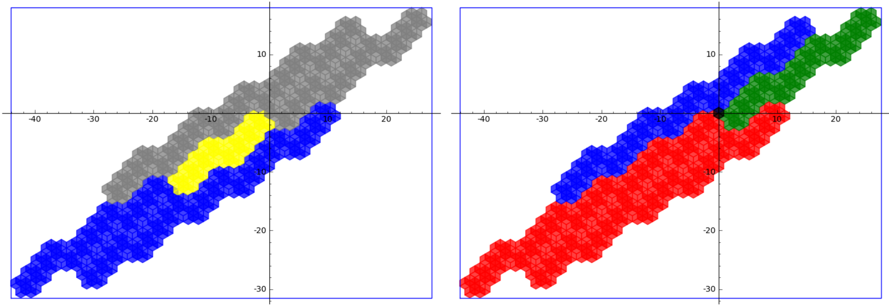

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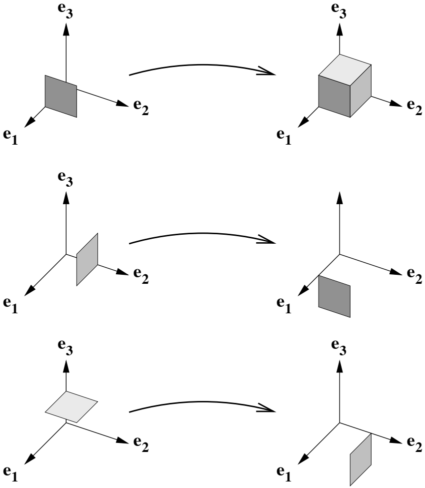
Construction par substitutions généralisées

$$\sigma: \quad \begin{array}{l} 1 \longrightarrow 12 \\ 2 \longrightarrow 13 \\ 3 \longrightarrow 1 \end{array}$$



Let $\mathcal{A}_d = \{1, 2, \ldots, d\}$ and $(e_1, e_2, \ldots, e_d)$ be the canonical base of $\mathbb{R}^d$. We consider $\mathfrak{F}$ be the vector space of mappings from $\mathbb{Z}^d \times \mathcal{A}_d$ to $\mathbb{R}$ that takes everywhere zero value except for a finite set.

Formalization of the Freeman chain-code

Let $\mathcal{A}_d = \{1, 2, \ldots, d\}$ and $(e_1, e_2, \ldots, e_d)$ be the canonical base of $\mathbb{R}^d$. We consider $\mathfrak{F}$ be the vector space of mappings from $\mathbb{Z}^d \times \mathcal{A}_d$ to $\mathbb{R}$ that takes everywhere zero value except for a finite set.
Let $[\vec{x}, i]$ be the element of $\mathfrak{F}$ that takes value 1 at $(\vec{x}, i)$ and 0 elsewhere.

$$\pi_d : \mathcal{A}_d^* \longrightarrow \mathfrak{F}$$

$$\pi_d(w) = \sum_{w=p \cdot i \cdot s} [\vec{p}, i]$$

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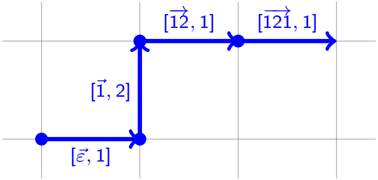
$\pi_d(1211) \to$ 

The E_1 operator

The 1-*dimensional geometric realization* $E_1(\sigma)$ of a word morphism σ is the linear mapping defined on $\mathfrak{F}$ such that :

$$\begin{tikzcd}
\mathcal{A}_d^* \arrow[r, "\sigma"] \arrow[d, "\pi_d"] & \mathcal{A}_d^* \arrow[d, "\pi_d"] \\
\mathfrak{F} \arrow[r, "E_1(\sigma)"] & \mathfrak{F}
\end{tikzcd}$$

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$$E_1(\sigma)[\vec{x}, e_i] := \sum_{\substack{uj \text{ prefix of } \sigma(i) \\ u \in \mathcal{A}_d^*, j \in \mathcal{A}_d}} \left[M_\sigma \vec{x} + \vec{u}, e_j\right].$$

We consider $\mathfrak{F}^*$ the dual space of $\mathfrak{F}$ and the linear form :

$$\langle [\vec{y}, j], [\vec{x}, i]^* \rangle \overset{\text{def.}}{=} \begin{cases} 1 \text{ if } \vec{x} = \vec{y} \text{ and } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

The dual operator E_1^* of E_1 is given by

$$\langle E_1(\sigma)[\vec{y}, j] \, , \, [\vec{x}, i]^* \rangle = \langle [\vec{y}, j] \, , \, E_1^*(\sigma)[\vec{x}, i]^* \rangle \, .$$

In the case where M_σ is unimodular

$$E_1^*(\sigma)[\vec{x}, i]^* := \sum_{j \in \mathcal{A}} \quad \sum_{ui \text{ prefix of } \sigma(j)} \left[M_\sigma^{-1} \left(\vec{x} - \vec{u} \right), j \right]^* .$$

We represent an element $[\vec{x}, i]^*$ as :

$$[\vec{x}, i]^* \longrightarrow \{\vec{x} + e_i + \sum_{i \neq j} \lambda e_j \mid \lambda \in [0, 1]\}.$$

Geometrical representation of $\mathfrak{F}^*$

We represent an element $[\vec{x}, i]^*$ as :

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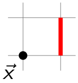
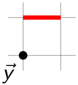
Examples :

- $d = 2$



$[\vec{x}, 1]^*$



$[\vec{y}, 2]^*$

We represent an element $[\vec{x}, i]^*$ as :

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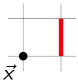

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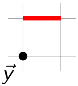
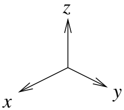
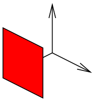


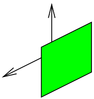
$[\vec{y}, 2]^*$

- $d = 3$

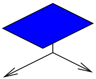




$[\vec{0}, 1]^*$



$[\vec{0}, 2]^*$

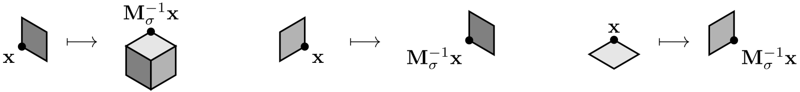


$[\vec{0}, 3]^*$

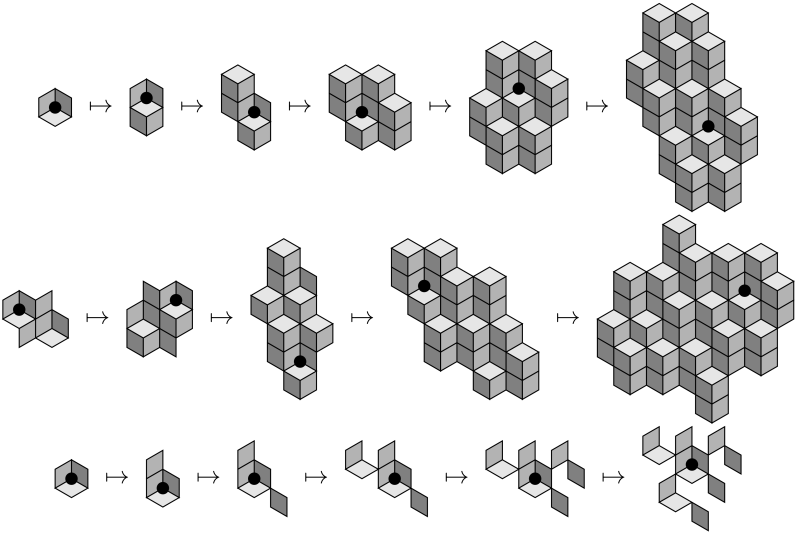
Exemple

$$\text{Soit } \sigma : \begin{cases} 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 1 \end{cases}$$

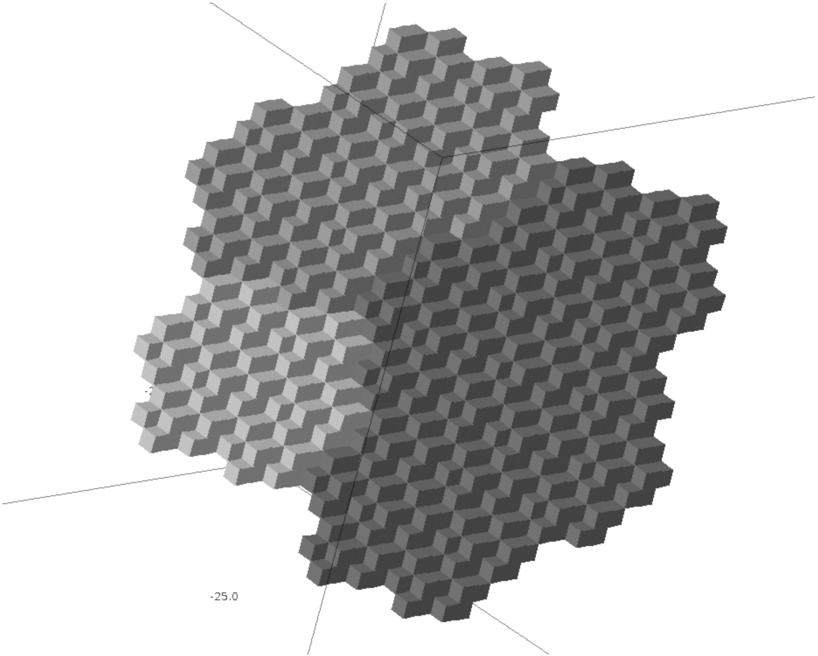
$$\begin{aligned}
E_1^*(\sigma)([x,1]^*) &= M_\sigma^{-1}x + \{[(1,0,-1),1]^* + [(0,1,-2),2]^* + [(0,0,0),3]^*\} \\
E_1^*(\sigma)([x,2]^*) &= M_\sigma^{-1}x + \{[(0,0,0),1]^*\} \\
E_1^*(\sigma)([x,3]^*) &= M_\sigma^{-1}x + \{[(0,0,0),2]^*\}
\end{aligned}$$



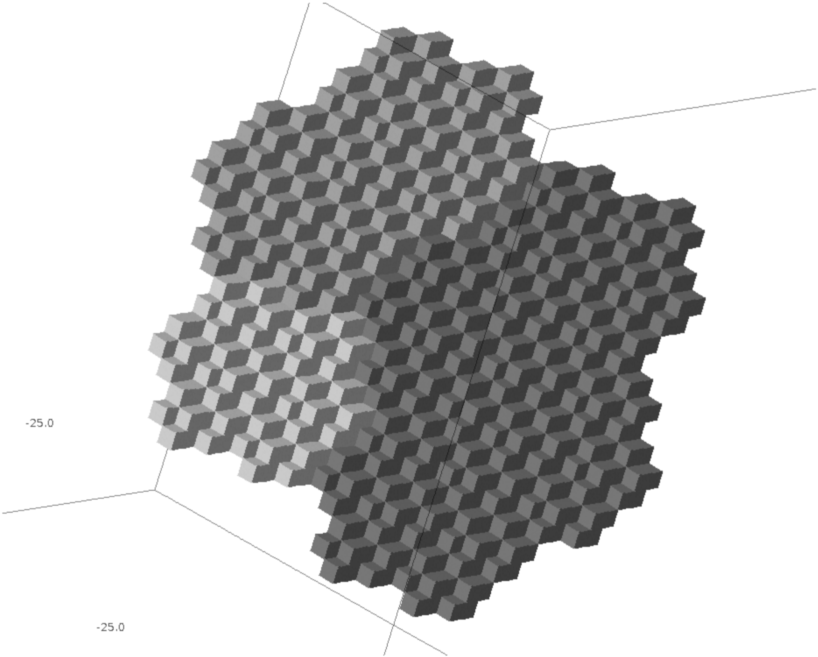
Exemple



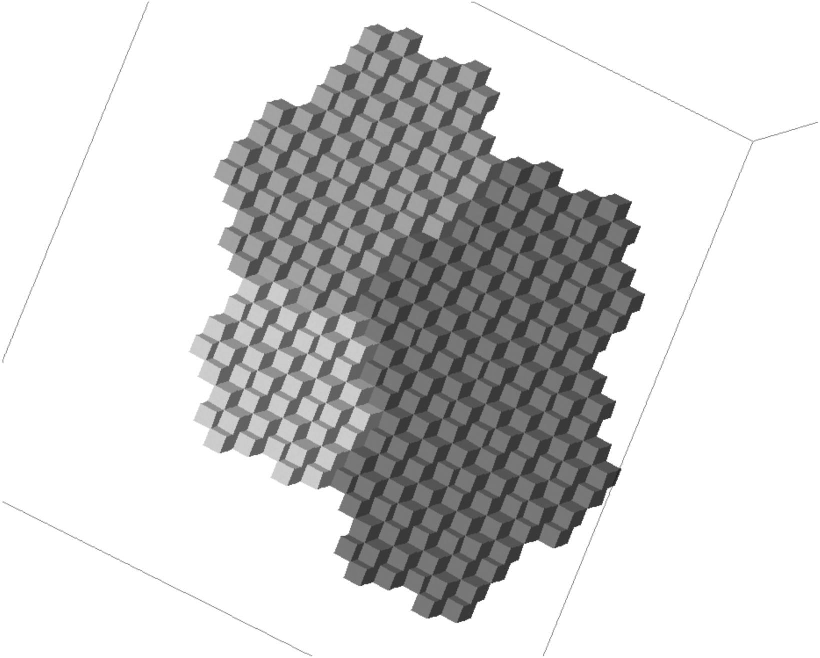
Exemple



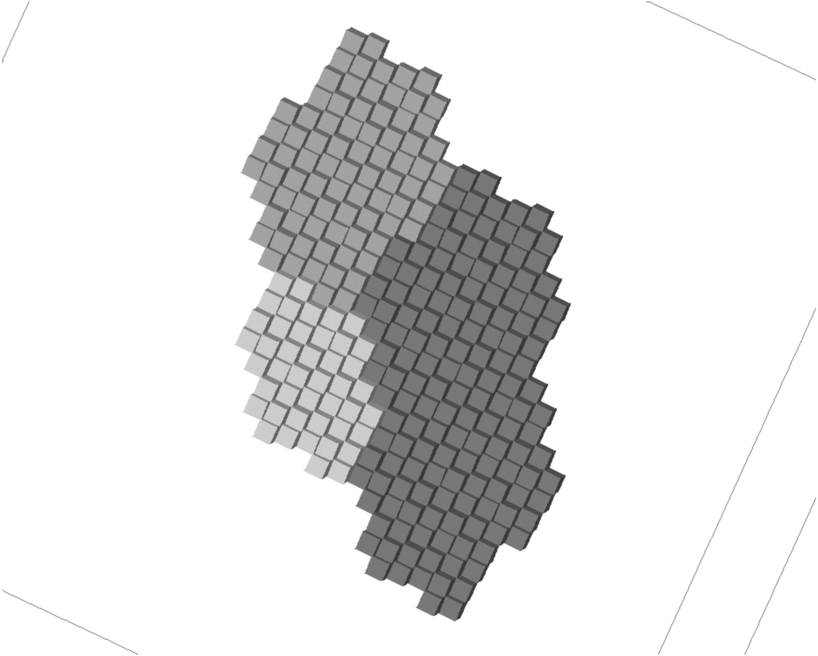
Exemple



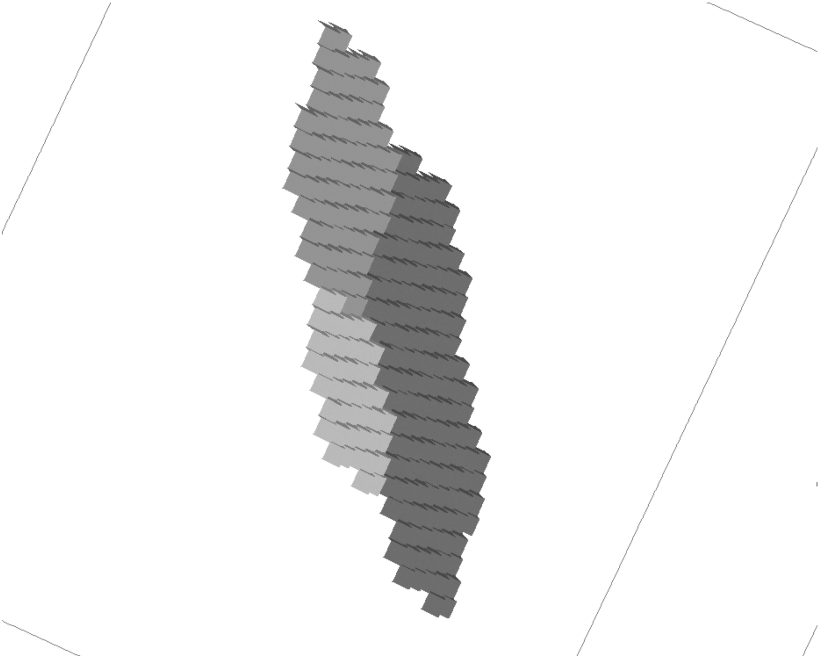
Exemple



Exemple



Exemple



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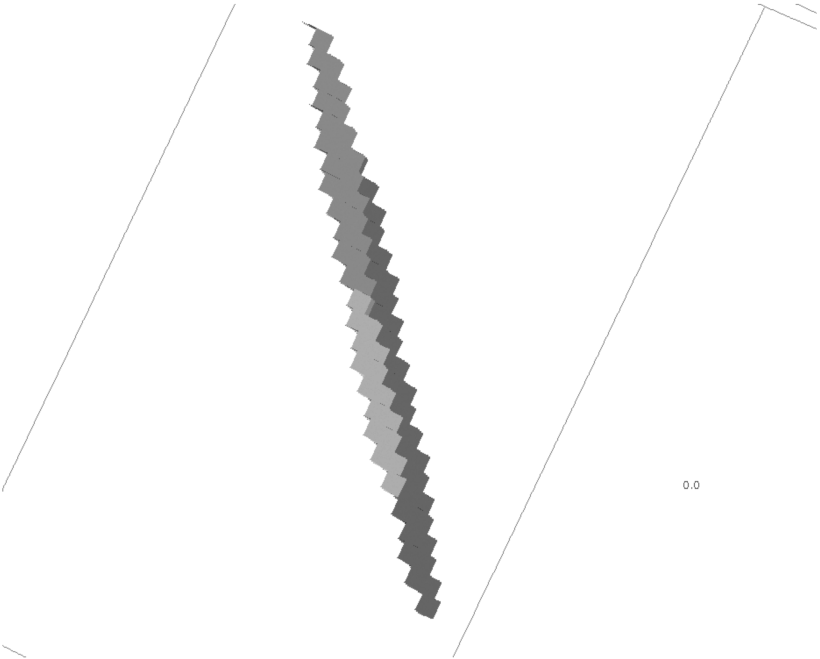
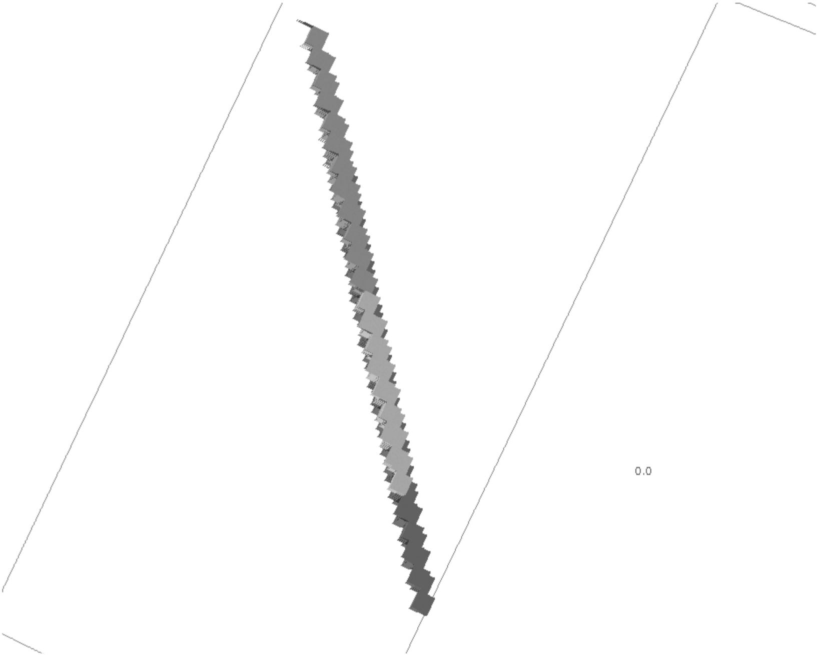
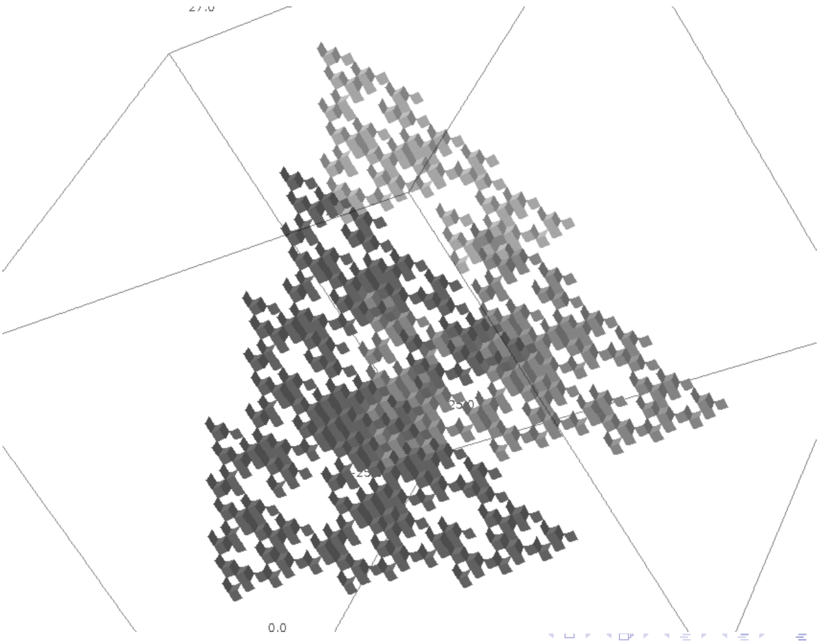
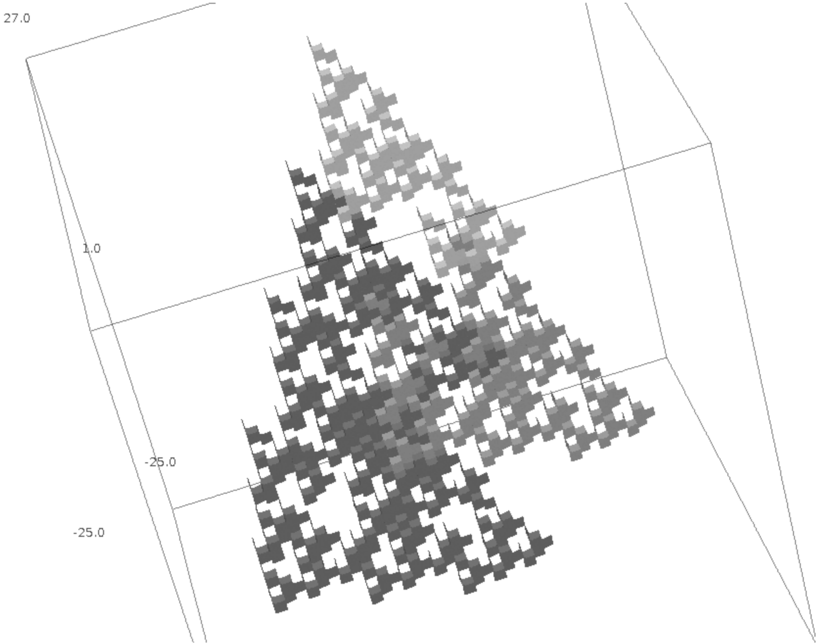


Exemple

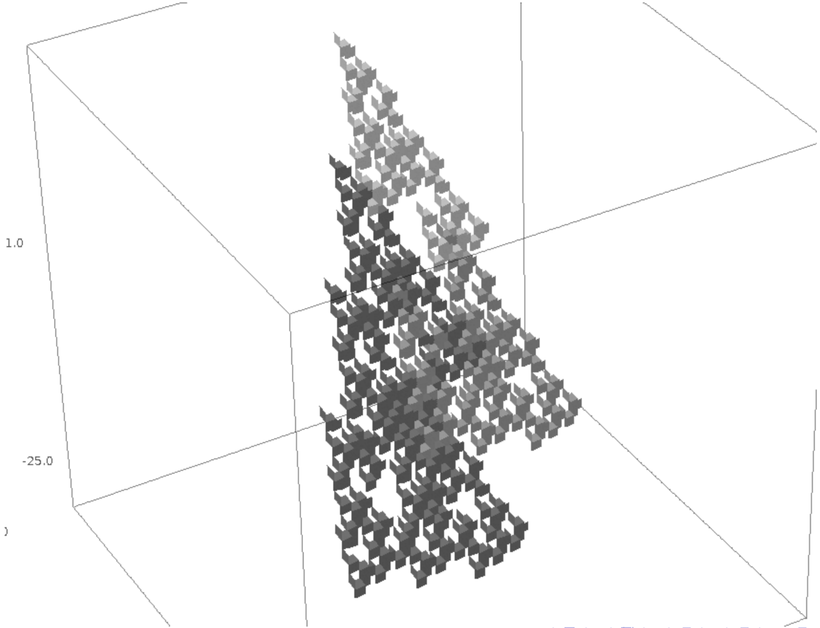



0.0

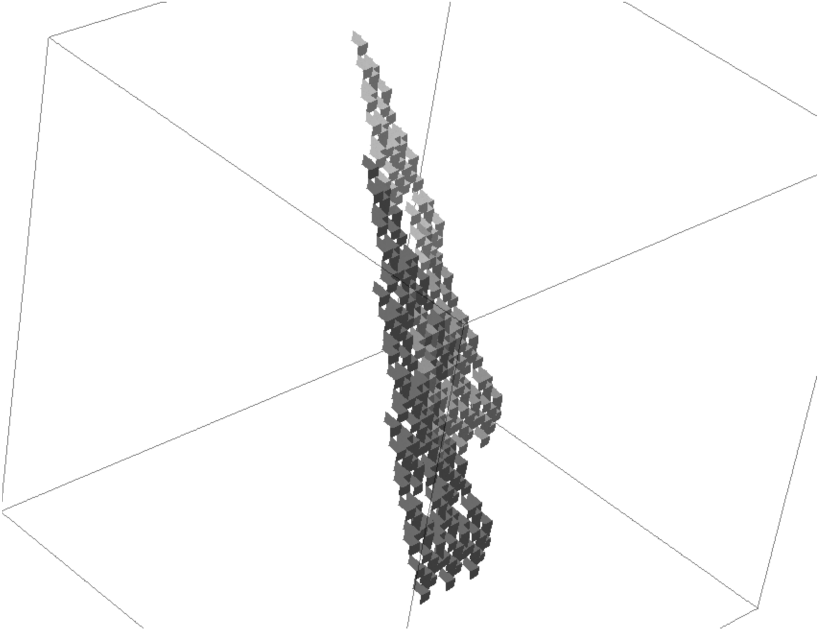
Exemple



Exemple



Exemple



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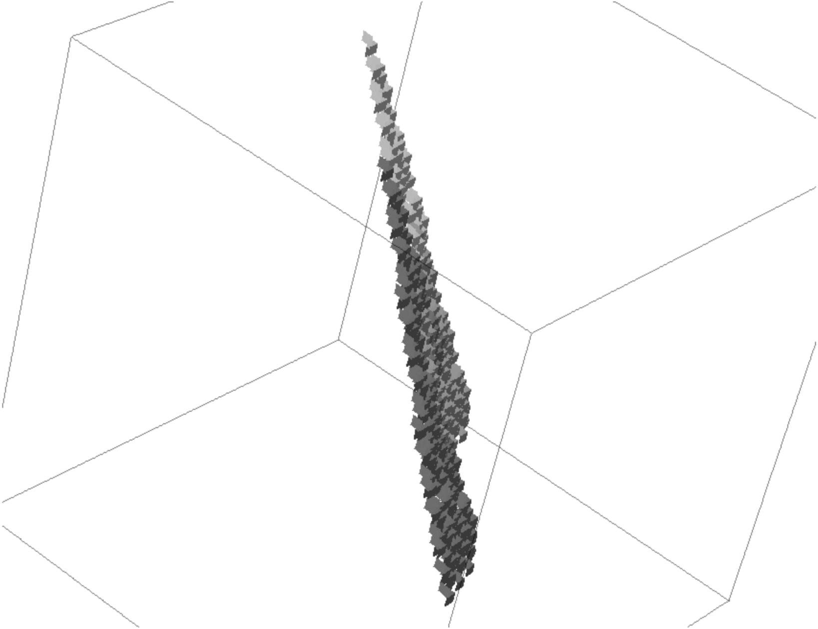
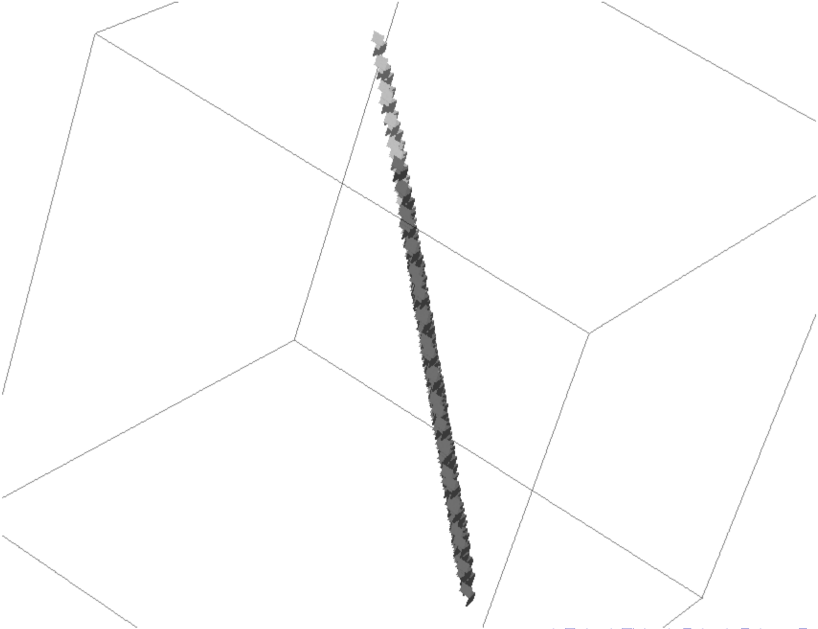


Exemple



Soit $\mathfrak{G}_{\vec{v}} = \{[\vec{x}, i]^* \in \mathfrak{F}^* \mid$ le plan normal à $\vec{v}$ intersects the segment $[\vec{x}, \vec{x} + e_i]\}$

Theorem (Arnoux, Ito, 2001)

Soit σ un morphisme primitif et unimodulaire, alors

$$E_1^*(\sigma)(\mathfrak{G}_{\vec{v}}) = \mathfrak{G}_{{}^t M_\sigma \vec{v}}$$

De plus, deux éléments distincts $(\vec{x}, e_i^*), (\vec{y}, e_j^*)$ ont des images disjointes par $E_1^*(\sigma)$.

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- $P \subset \mathfrak{G}_{\vec{v}} \implies E_1^*(\sigma)(P) \subset \mathfrak{G}_{{}^tM_\sigma \vec{v}}$.

Action sur les plans discrets

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- Pour tout $\vec{v} \in \mathbb{R}_+^3$ on a  $\subset \mathfrak{G}_{\vec{v}}$.

Question

Étant donné un vecteur $\vec{v}$, comment générer $(\sigma_i)_{i \geq 0}$ telle que chaque M_{σ_i} soit primitif, unimodulaire et

$$\lim_{n \to \infty} {}^tM_{\sigma_n} {}^tM_{\sigma_{n-1}} \cdots {}^tM_{\sigma_0}(0, 0, 1) = \vec{v}.$$

Computation of $[z_0; z_1, z_2, \ldots]$ from $\alpha = b/a$.

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- Let $u_0 = b$ and $u_1 = a$,
- For $i \geq 0$, (while $u_{i+1} > 0$)
  let $z_i = \left\lfloor \frac{u_i}{u_{i+1}} \right\rfloor$ and set $u_{i+2} = u_i - z_i u_{i+1}$.

Euclid's algorithm

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First steps :

$$u_2 = u_0 - \left\lfloor \frac{u_0}{u_1} \right\rfloor u_1,$$

$$u_3 = u_1 - \left\lfloor \frac{u_1}{u_2} \right\rfloor u_2,$$

$$\vdots$$

Computation of $[z_0; z_1, z_2, \dots]$ from $\alpha = b/a = b_0/a_0$.

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} -z_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \text{ where } z_n = \left\lfloor \frac{b_n}{a_n} \right\rfloor$$

Euclid's algorithm, matrix form

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If $a_n \neq 0$ let $M_{z_n} = \begin{bmatrix} 0 & 1 \\ 1 & z_n \end{bmatrix}$, otherwise $M_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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$$\begin{bmatrix} a \\ b \end{bmatrix} = M_{z_1} M_{z_2} \cdots M_{z_n} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

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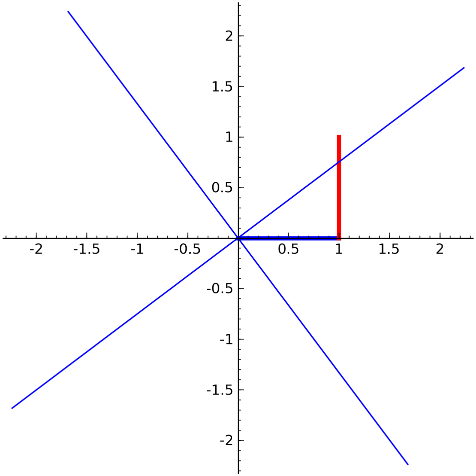
$$\begin{bmatrix} a \\ b \end{bmatrix} = \lim_{n \to \infty} M_{z_1} M_{z_2} \cdots M_{z_n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(a,b) = (\pi, \sqrt{3}),\ \sqrt{3}/\pi = [0; 1, 1, 4, 2, 1, 2, 3, 7, 3, \dots],$$

$E_1(\tau_0)(\vec{0}, e_2)$

$E_1^*(\tau_0)(\vec{0}, e_2^*)$

$$\tau_n = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12^n \end{cases}$$



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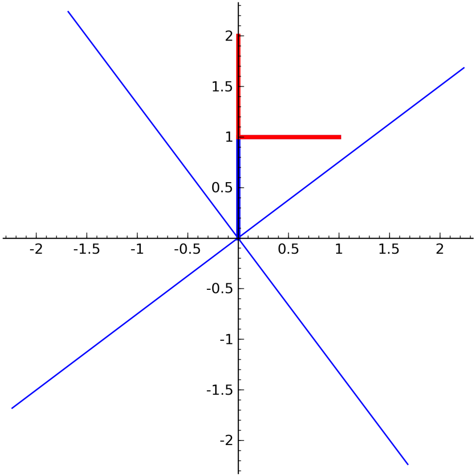

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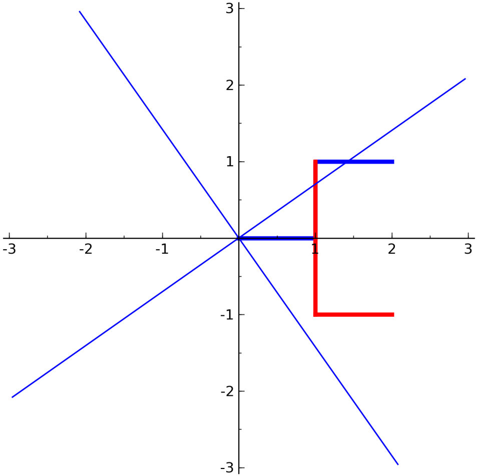

Example in dimension 2

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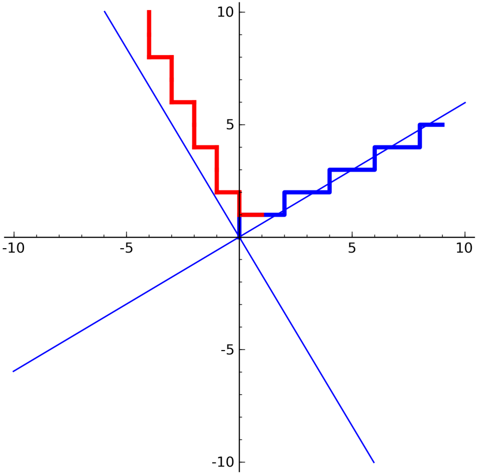

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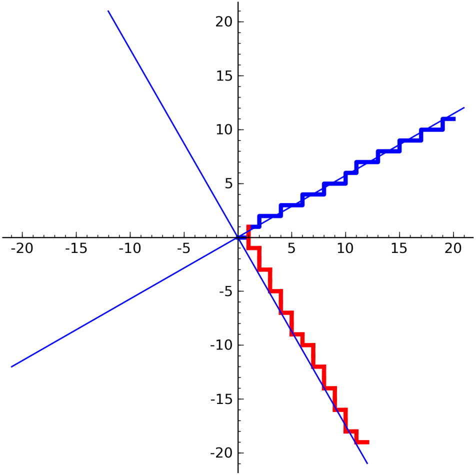

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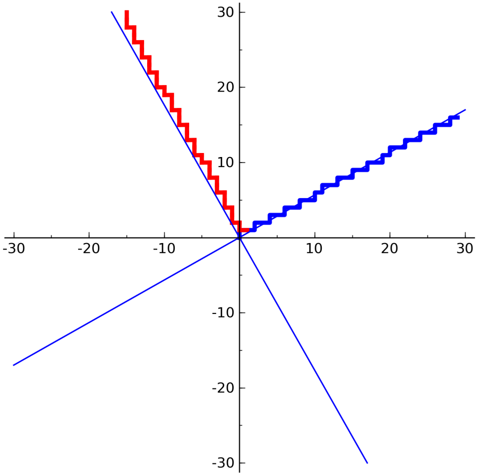

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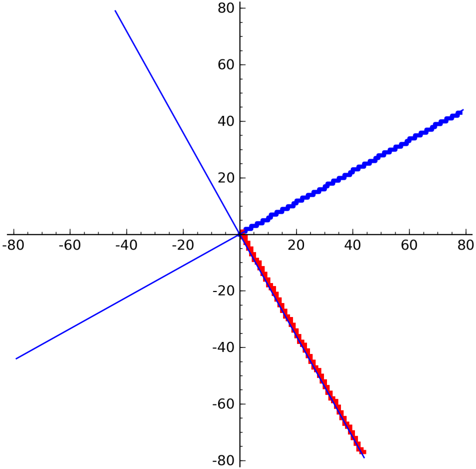

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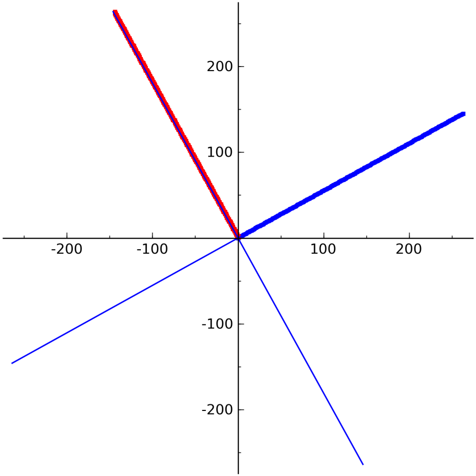

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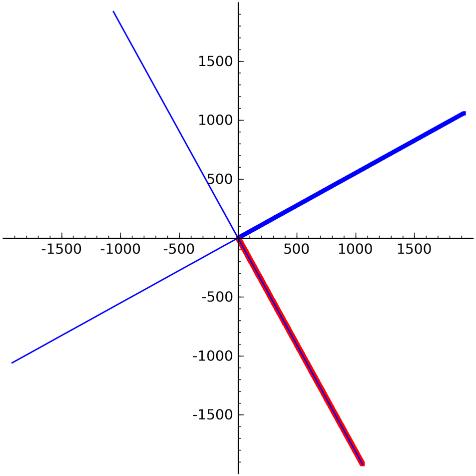

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$$E_1^*(\tau_0) \circ E_1^*(\tau_1) \circ E_1^*(\tau_1) \circ E_1^*(\tau_4) \circ E_1^*(\tau_2) \circ E_1^*(\tau_1) \circ E_1^*(\tau_2) \circ E_1^*(\tau_3) \circ E_1^*(\tau_7) \circ E_1^*(\tau_3)(\vec{0}, e$$

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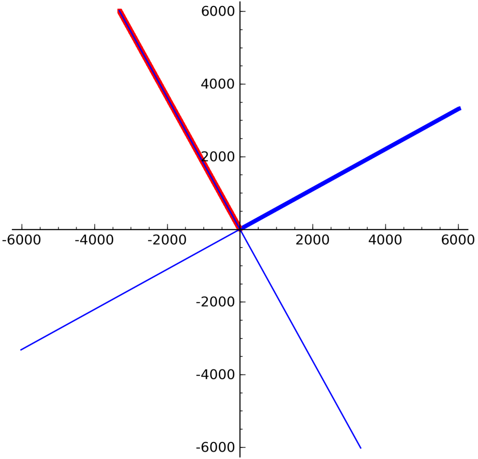

Theorem (Berthé, de Luca, Reutenauer, 2007)

The geometrical representation of

$$E_1^*(\tau_{z_0}) \circ E_1^*(\tau_{z_1}) \circ \cdots \circ E_1^*(\tau_{z_n})(\vec{0}, e_2^*)$$

codes the Christoffel word of slope p_n/q_n *where* $q_n/p_n = [z_0; z_1, \ldots, z_n]$.

Quelques algorithes de fractions continues unimodulaires. Soit $(a, b, c) \in \mathbb{R}^3_+$ avec $a \leq b \leq c$,

- Jacobi-Perron :
$$(a, b, c) \mapsto \left(b - \left\lfloor \frac{b}{a} \right\rfloor, c - \left\lfloor \frac{c}{a} \right\rfloor, a\right)$$
- Brun :
$$(a, b, c) \mapsto (a, b, c - b)$$
- Poincarré :
$$(a, b, c) \mapsto (a, b - a, c - b)$$
- Selmer :
$$(a, b, c) \mapsto (a, b, c - a)$$
- Fully substractive :
$$(a, b, c) \mapsto (a, b - a, c - a)$$

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$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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$${}^tM_1^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^tM_2^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^tM_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$s_1 := \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \quad s_2 := \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12 \\ 3 \mapsto 32 \end{cases} \quad s_3 := \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 13 \\ 3 \mapsto 23 \end{cases}$$

Étant donné un vecteur $v \in \mathbb{R}^3_+$ avec $v_1 \leq v_2 \leq v_3$, on pose $v^{(0)} = v$ et

$$v^{(n+1)} = M^{(n)} v^{(n)},$$

Théorème

Les points générés par le E_1^* de fully substractive est forment un sous-ensemble de la construction *translation* + *union*.

$$\mathcal{P}_n = \{x \in \mathbb{Z}^3 \mid [x, i]^* \in E_1^*(s^{(0)}) E_1^*(s^{(1)}) \cdots E_1^*(s^{(n)})($$  $$)\}$$

Théorème d'équivalence

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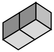
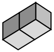
Sketch de la preuve,
Pour tout $i \in \{1,2,3\}$ on a

$$E_1^*(s_i)($$  $$) =$$ 

Ainsi,

$$E_1^*(s^{(0)})E_1^*(s^{(1)}) \cdots E_1^*(s^{(n)})($$  $$) = E_1^*(s^{(0)})E_1^*(s^{(1)}) \cdots E_1^*(s^{(n-1)})($$  $$)$$

$$= E_1^*(s^{(0)})E_1^*(s^{(1)}) \cdots E_1^*(s^{(n-1)})($$  $$) + E_1^*(s^{(0)})E_1^*(s^{(1)}) \cdots E_1^*(s^{(n-1)})($$  $$)$$

Génération de la totalité du plan

Notation : $\Sigma_i = E_1^*(s_i)$.

Théorème (Jolivet, 2012)

Soit $(i_n)_{n\geq 0}$ avec chaque $i_n \in \{1,2,3\}$ tel que pour tout $n \geq 0$, il existe $N > n$ avec $i_N = 3$,

$$\Sigma_{i_0}\Sigma_{i_1}\cdots\Sigma_{i_n}(\quad)$$

contient une boule centrée en 0 arbitrairement grande lorsque n tends vers l'infini.

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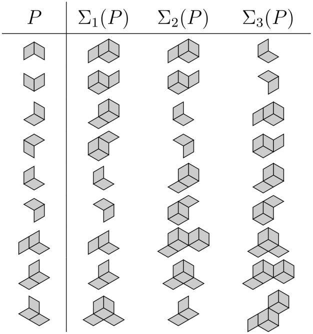
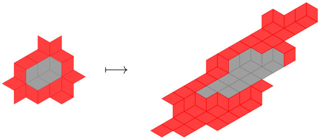
Sketch de la preuve





Construction par composantes connexes

Définition

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- On appelle $\mathcal{T}(v, h)$ la composante 2-connexe de $\vec{0}$ dans $P'(v, 0, h)$.
- $\mathcal{B}(v, h) = \{x \in \mathcal{P}'(v, 0, \Omega_2(v, 0)) \setminus \mathcal{T}(v, h) \,|\, x$ est 2-voisin d'un point de $\mathcal{T}(v, h)\}$.

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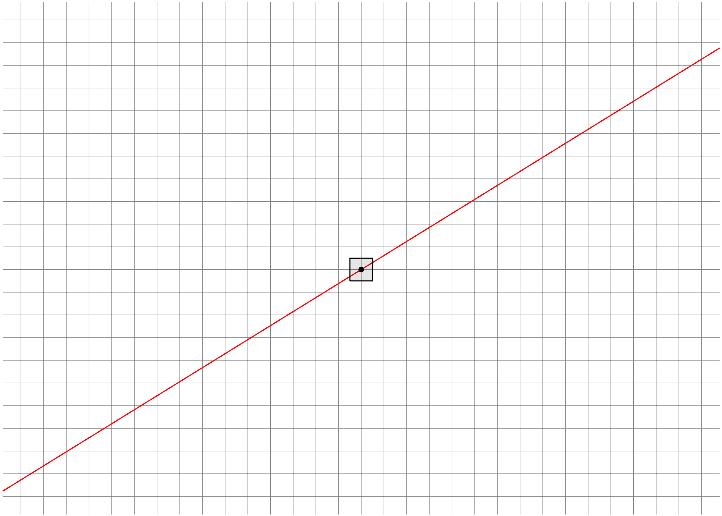
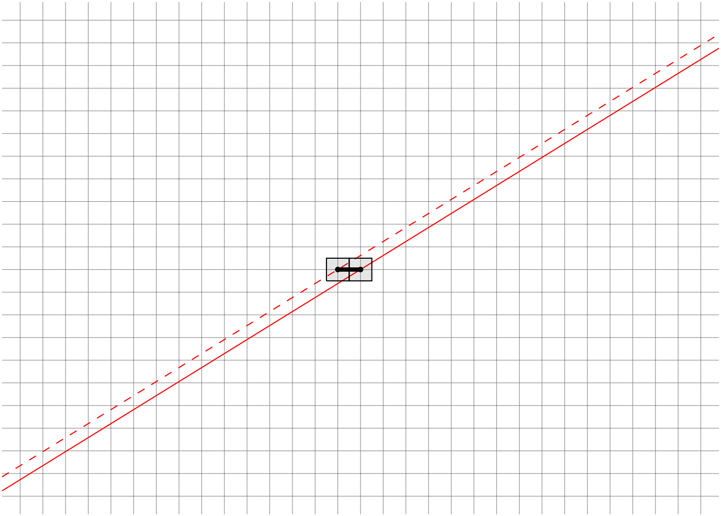
- $\mathcal{B}(v, h) = \{x \in \mathcal{P}'(v, 0, \Omega_2(v, 0)) \setminus \mathcal{T}(v, h) \mid x$ est 2-voisin d'un point de $\mathcal{T}(v, h)\}$.
- $h_0 = 0$,
- $h_{i+1} = \min\{\langle X, v\rangle \mid X \in \mathcal{B}(v, h_i)\}$

On s'intéresse à la suite de *morceaux de plans* $(\mathcal{T}(v, h_i))_{i \geq 0}$.

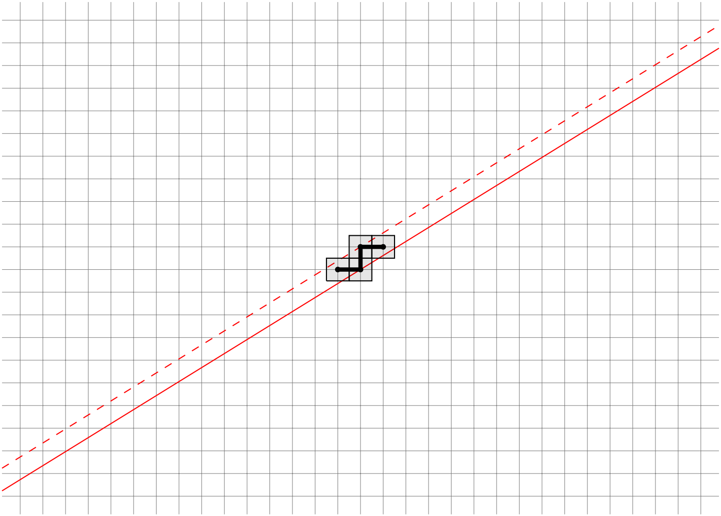
Exemple 2D



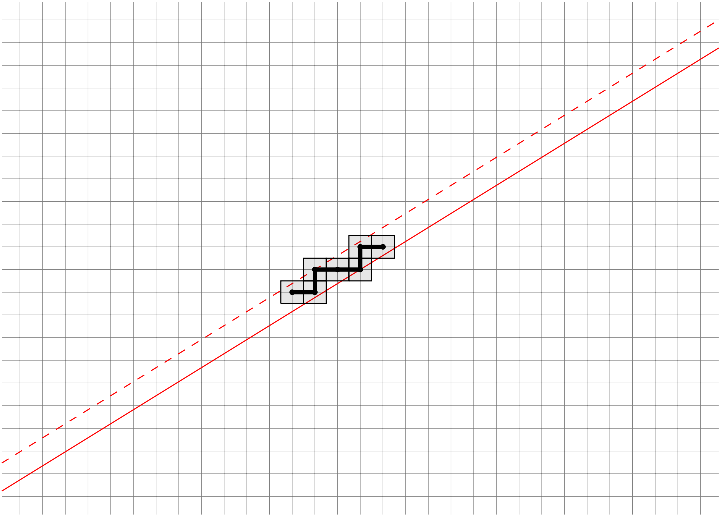
Exemple 2D

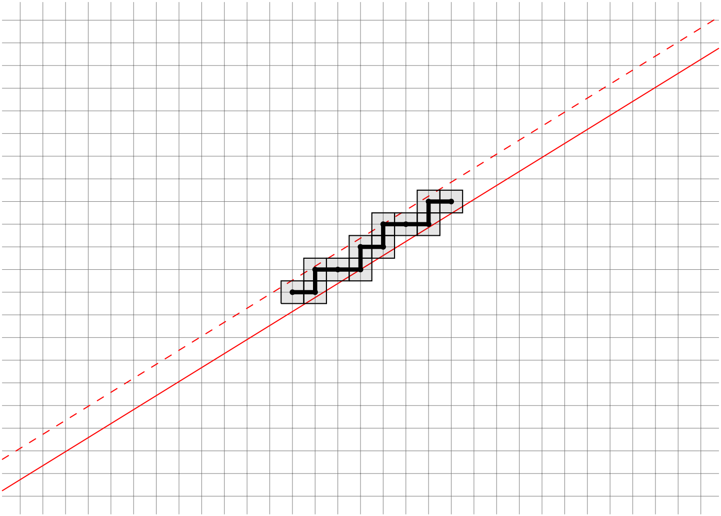


Exemple 2D



Exemple 2D





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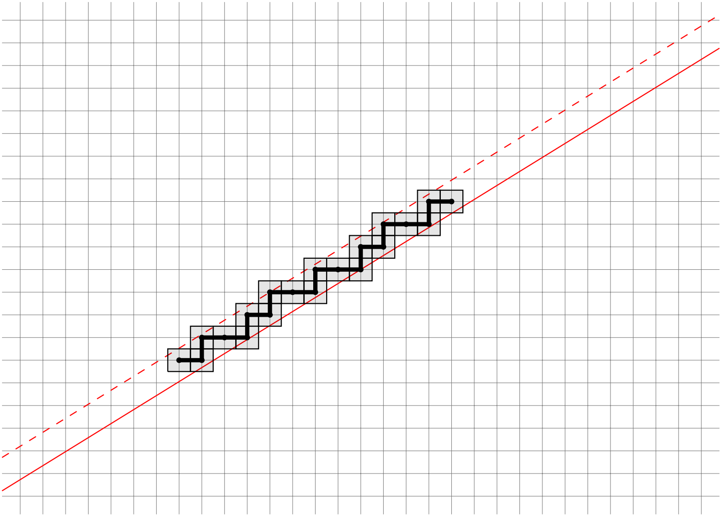
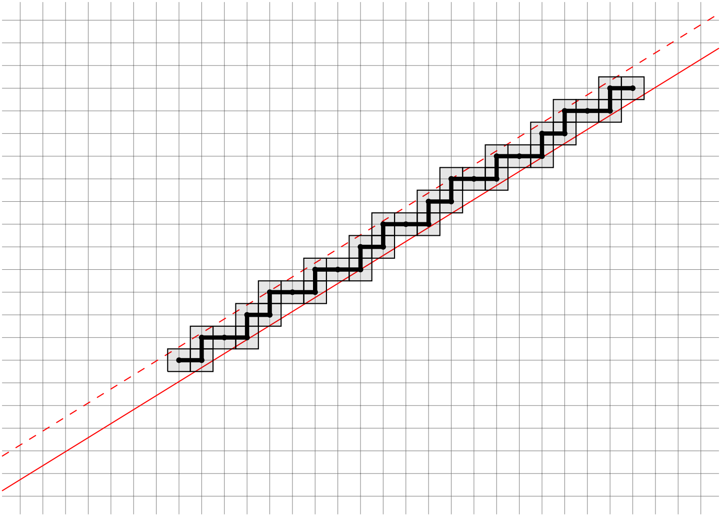
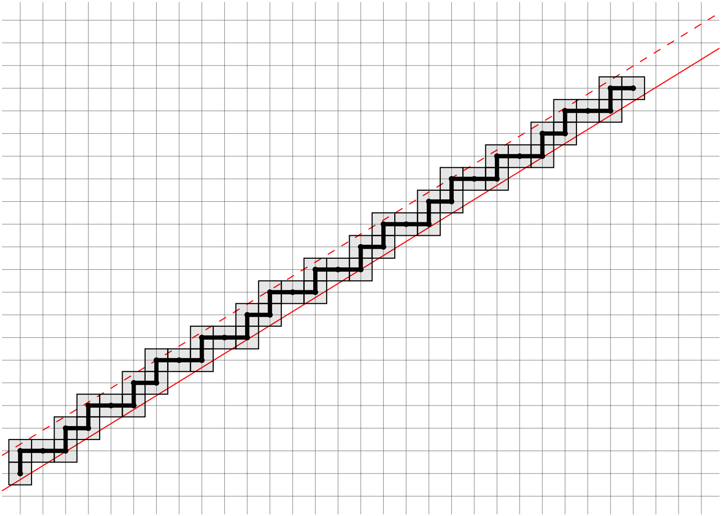
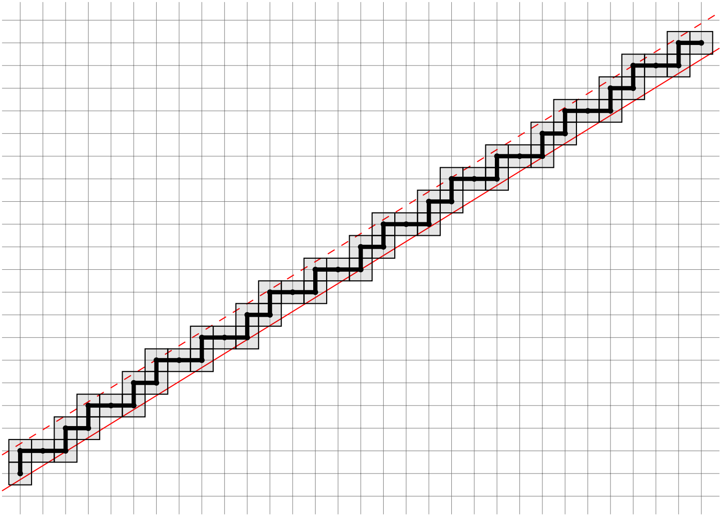




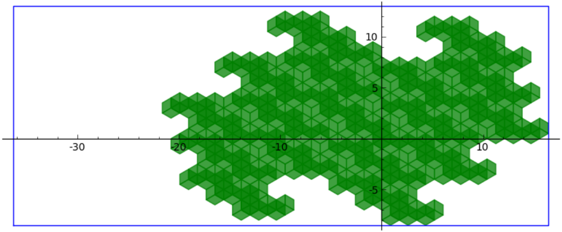
Exemple 2D

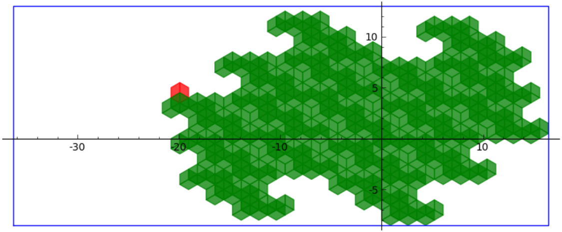


Exemple 2D

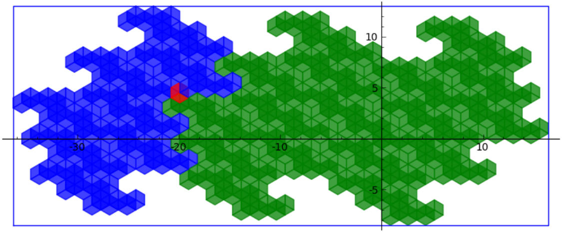


Exemple 3D





10
5
-30
-20
-10
10
-5


10
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-5
-30
-20
-10
10

Équivalence des constructions

Théorème (Berthé, 2012 (?))

$\mathcal{T}(v, h_n) = \{x \in \mathbb{Z}^3 \mid [x, i]^* \in E_1^*(s^{(0)})E_1^*(s^{(1)}) \cdots E_1^*(s^{(n)})($  $)\}$

MERCI !