Two linear-time algorithms for computing the minimum length polygon of a digital contour

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Fonds de recherche sur la nature et les technologies Québec

Polyominoes

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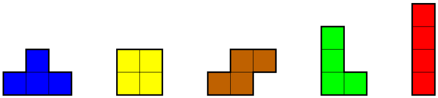

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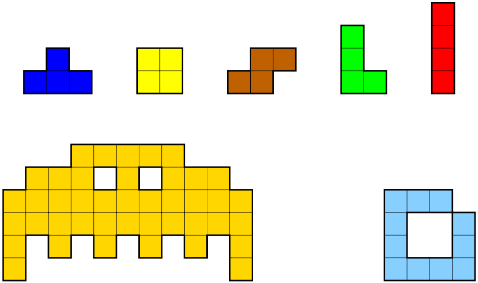

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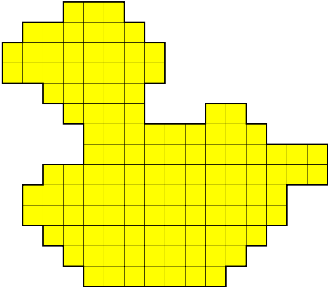

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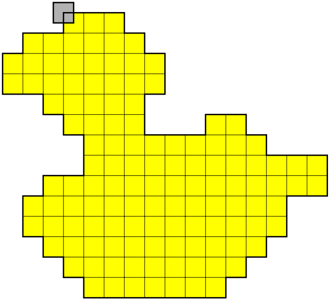

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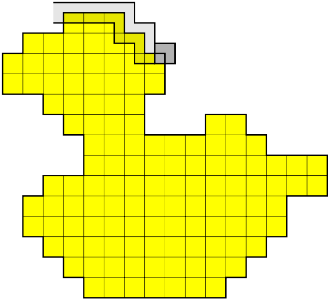

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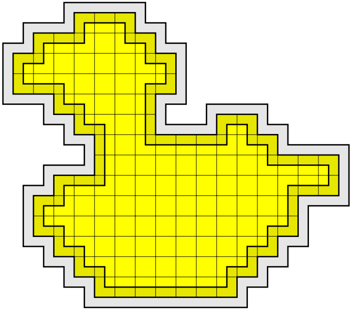

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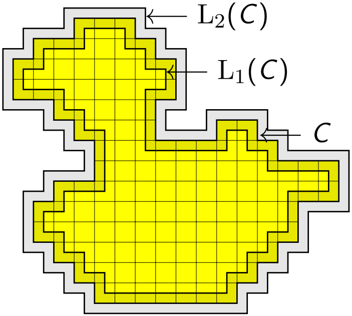

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Given a digital contour C, its *inner* (resp. *outer*) *polygon* $\mathrm{L}_1(C)$ (resp. $\mathrm{L}_2(C)$) is the erosion (resp. dilatation) of the body of $I(C)$ by the open unit square centrer on $(0, 0)$.

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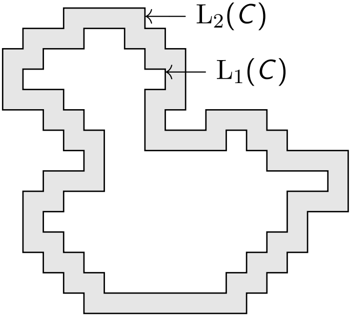

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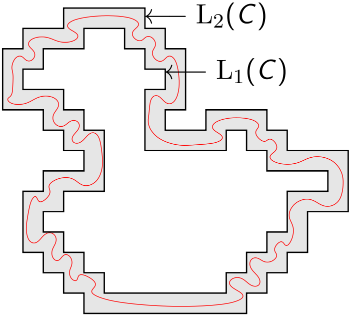

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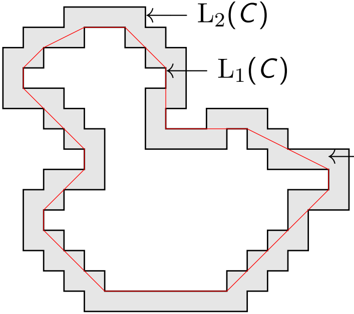

Definition

The *minimum length polygon* of C is a subset $P \in \mathbb{R}^2$ such that,

$$P = \underset{A \in \mathcal{A},\, \mathrm{L}_1(C) \subseteq A,\ \partial A \subset \mathrm{L}_2(C) \setminus \mathrm{L}_1(C)^\circ}{\arg\min} \mathrm{Per}(A)$$

where $\mathcal{A}$ is the family of simply connected compact sets of $\mathbb{R}^2$.

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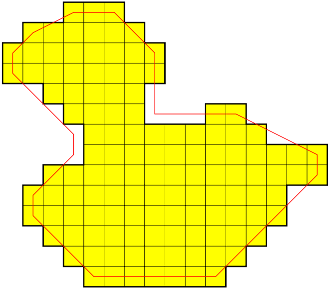

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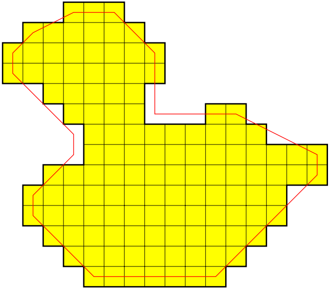

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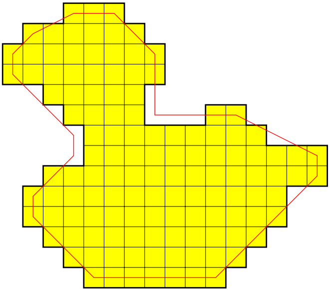

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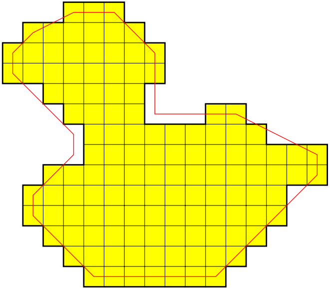

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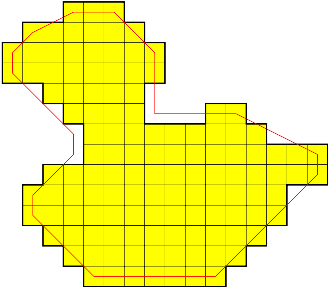

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- reversible*.

Relative Convex Hull

Definition

Let $U \in \mathbb{R}^2$ and $V \subseteq U$. The set V is said to be *U-convex* if for every $x, y \in V$

$$\overline{xy} \subseteq U \implies \overline{xy} \subseteq V.$$

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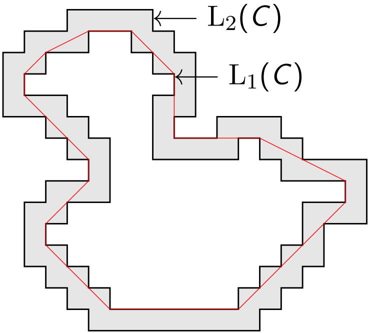
- Every set $U \subseteq \mathbb{R}^2$ is U-convex.
- The usual convexity is the $\mathbb{R}^2$-convexity.

Set-Theoretic MLP

Theorem (Sloboda, Stoer 1994)

The MLP of a digital contour C, is the $\mathrm{L}_2(C)$-convex hull of $\mathrm{L}_1(C)$.



The Arithmetic MLP

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Definition

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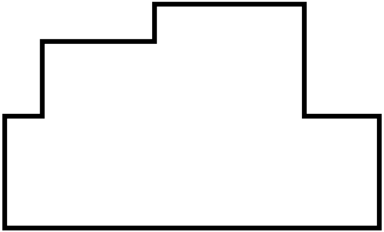

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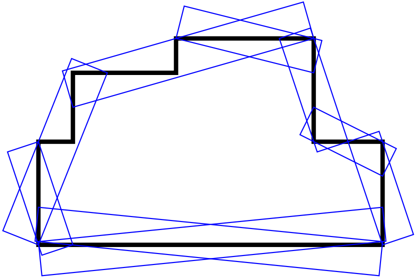

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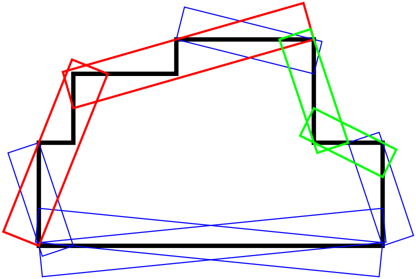

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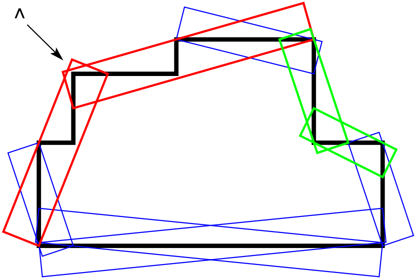

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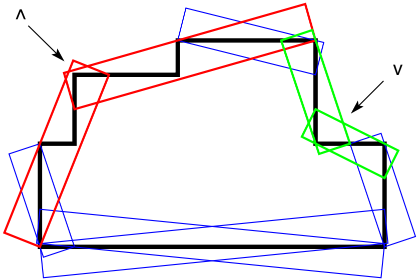

- $\wedge$ stands for a "convex turn",
- $\vee$ stands for a "concave turn".

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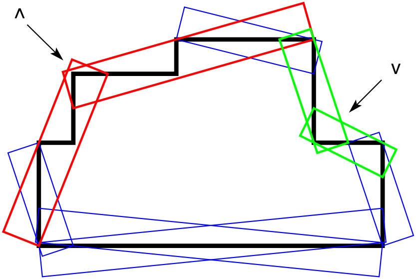
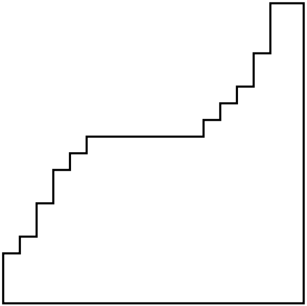


$\dots$	$\wedge \overline{M_i} \wedge \dots \wedge \overline{M_j} \wedge$	$\overline{M_{j+1}}$	$\vee \overline{M_{j+2}} \vee \dots \vee \overline{M_k} \vee$	$\overline{M_{k+1}}$	$\wedge \dots$
$\dots$	convex zone	inflexion	concave zone	inflexion	$\dots$

Tangential Cover and Convexity

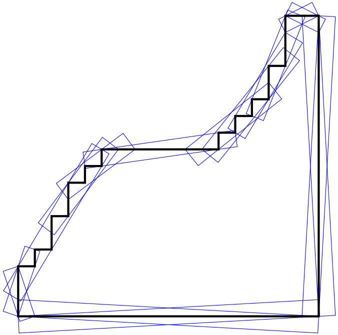
Theorem (Dorksen-Reiter, Debled-Rennesson 2006)

A digital contour is digitally convex iff every couple of consecutive MDSS of its tangential cover is made of $\wedge$-turns.



Definition

A digital contour C with tangential cover $M_1, M_2, \ldots, M_l$ is uniquely split into a sequence of closed connected sets with a single point overlap as follows :



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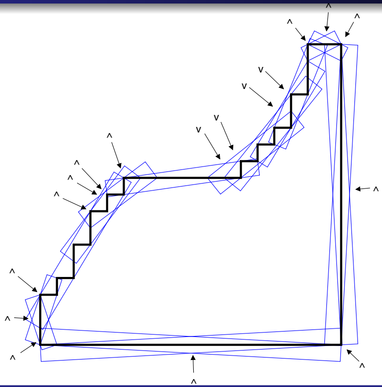

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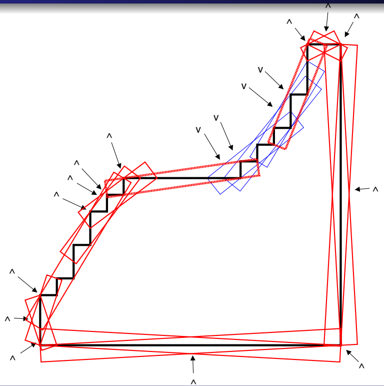
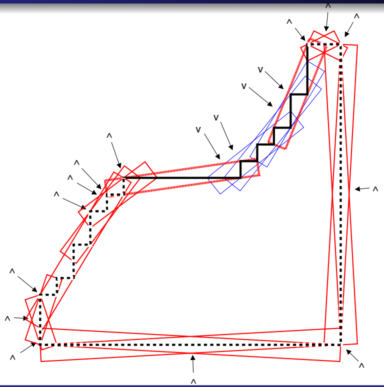


Definition

A *convex zone or* $(\wedge, \wedge)$*-zone* is an inextensible sequence of consecutive $\wedge$-turns from (M_i, M_{i+1}) to (M_j, M_{j+1}). If $i \neq j$, if starts at $U_l(M_i)$ and ends at $U_f(M_{j+1})$.

Convex, Concave and Inflexion Zones



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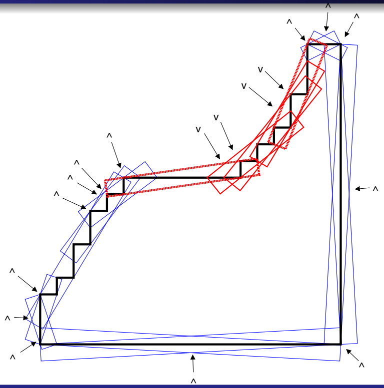

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A *concave zone or* $(\vee, \vee)$*-zone* is an inextensible sequence of consecutive $\vee$-turns from (M_i, M_{i+1}) to (M_j, M_{j+1}). If $i \neq j$, if starts at $L_l(M_i)$ and ends at $L_f(M_{j+1})$.

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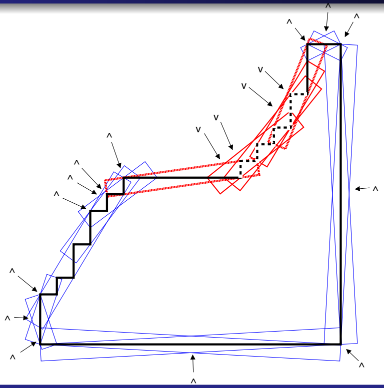

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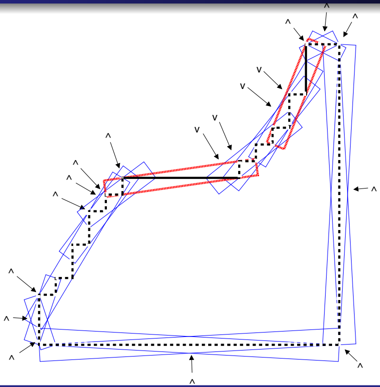
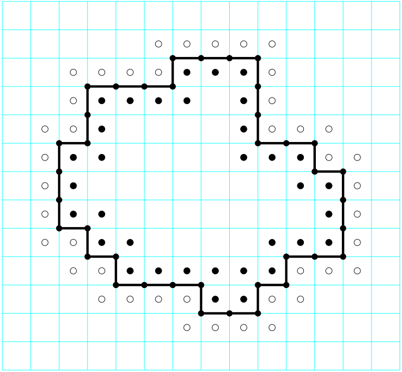


Definition

A *convex inflexion zone or* $(\wedge, \vee)$*-zone* is a $\wedge$-turn followed by a $\vee$-turn around a MDSS M_i. It starts at $U_f(M_i)$ and ends at $L_L(M_i)$. A *concave inflexion zone or* $(\vee, \wedge)$*-zone* is a $\vee$-turn followed by a $\wedge$-turn around a MDSS M_i. It starts at $L_f(M_i)$ and ends at $U_L(M_i)$.

Inside and Outside Pixels



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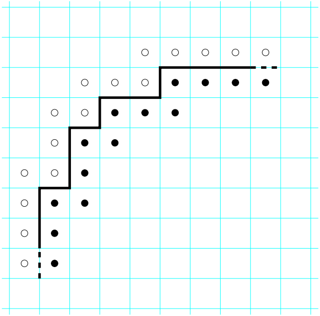

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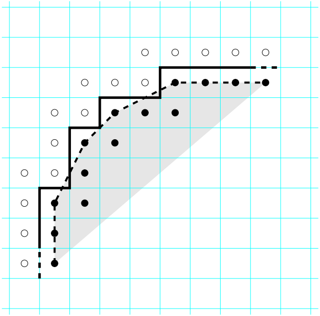

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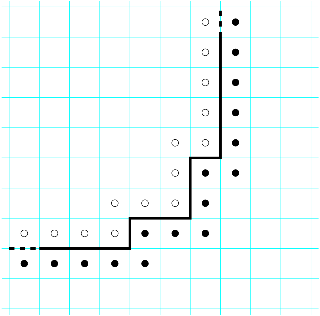

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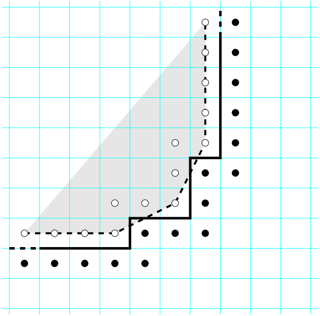

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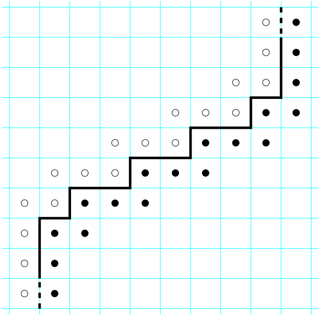

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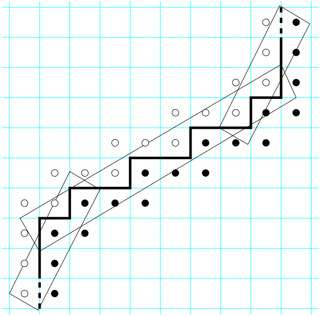

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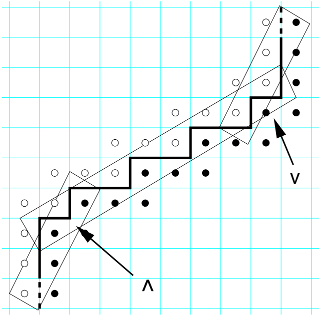

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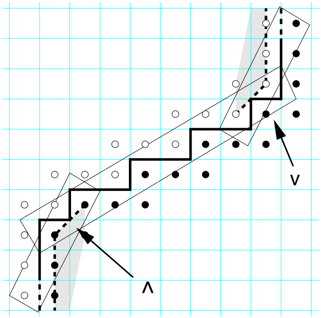

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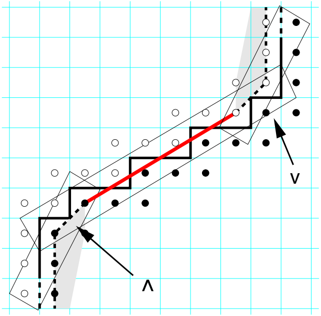

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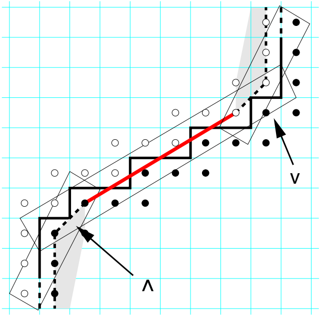

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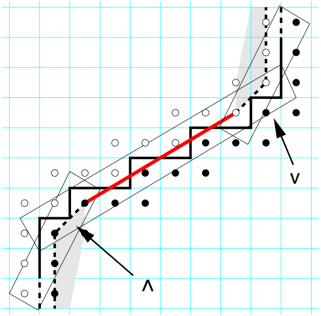

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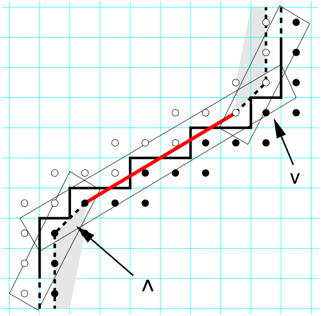

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Given a digital contour C, the AMLP of C is the MLP of C.

Proof: show that the AMLP of C is the convex hull of $\mathrm{L}_1(C)$ relatively to $\mathrm{L}_2(C)$.

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(2) Decompose $(M_i)_{i=1..N}$ is (α, β)-zones. $(\alpha, \beta \in \{\wedge, \vee\})$

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(2) is performed in $O(N)$ where $N \leq n$.

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Three steps to compute the AMLP if C:

(1) Compute $(M_i)_{i=1..N}$ the tangential cover of C;

(2) Decompose $(M_i)_{i=1..N}$ is (α, β)-zones. $(\alpha, \beta \in \{\wedge, \vee\})$

(3) For each (α, β)-zones, compute the associate part of the AMLP.

(1) is performed in $O(n)$ using Lachaud, Vialard and de Vieilleville 2007 (DGCI 2005).

(2) is performed in $O(N)$ where $N \leq n$.

(3) is performed in $O(n)$ using Melkman 1987 on each $(\wedge, \wedge)$ or $(\vee, \vee)$-zones; while each $(\wedge, \vee)$ or $(\vee, \wedge)$-zones is treated in constant time.

The Combinatorial MLP

- A *contour word* is the Freeman code of the border of a polyomino.

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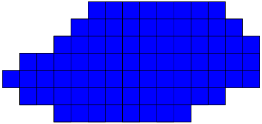
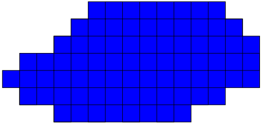


10100101010000000030303332232232222222212212

- A *contour word* is the Freeman code of the border of a polyomino.
- A *quadrant word* is an inextendable factor of a contour word over two letters.



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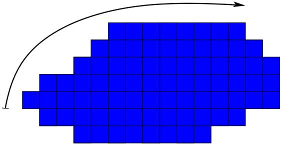

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Combinatorial View of Digital Convexity

- A *contour word* is the Freeman code of the border of a polyomino.
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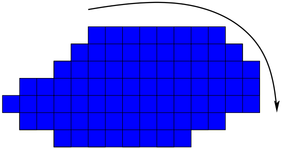

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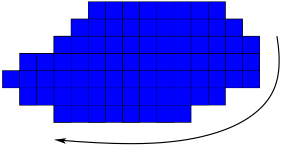

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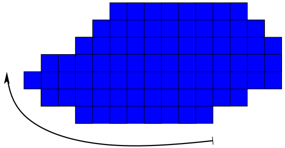

1010010101000000003030333223223222222221221 2

Combinatorial View of Digital Convexity

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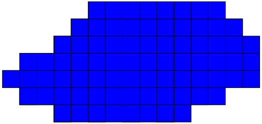
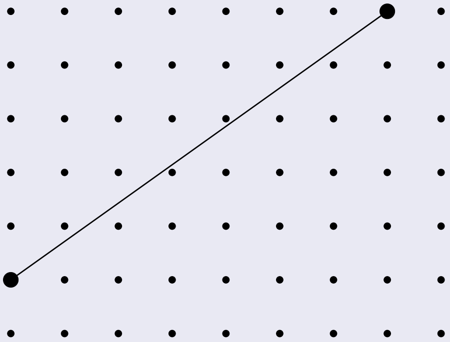


Theorem (Brlek, Lachaud, P., Reutenauer "DGCI 2008")

An hv-convex polyomino P is digitally if and only if the each of its quadrant words q_i is such that its factorization as decreasing Lyndon words $q_i = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ contains only Christoffel words. In such case, this factorization coincide with its Euclidian convex.

Christoffel Words

Definition (Borel and Laubie 1993)



Christoffel Words

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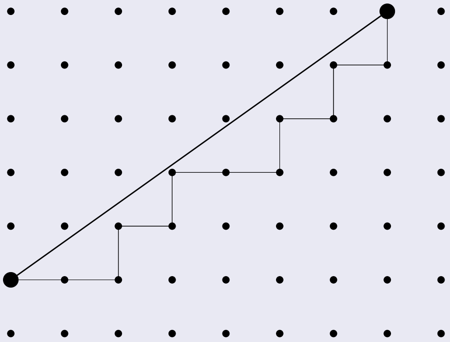
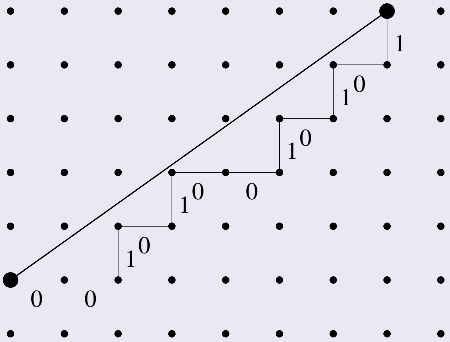


Definition (Borel and Laubie 1993)



$$\mathcal{C}_{5/7} = 001010010101.$$

Lyndon Words

Definition

A word w over an ordered alphabet is a Lyndon word if for all non-empty words u and v:

$$w = uv \implies w < vu.$$

(where $<$ denotes the lexicographic order.)

Theorem (Lyndon 1950)

Any word w over an ordered alphabet admits a unique factorization as decreasing Lyndon words.

Lyndon Words

Definition

A word w over an ordered alphabet is a Lyndon word if for all non-empty words u and v:

$$w = uv \implies w < vu.$$

(where $<$ denotes the lexicographic order.)

Theorem (Lyndon 1950)

Any word w over an ordered alphabet admits a unique factorization as decreasing Lyndon words.

Theorem (Duval 1983)

Given a word w of length n, its factorization as decreasing Lyndon words $w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ is computed in $O(n)$ using Duval's algorithm.

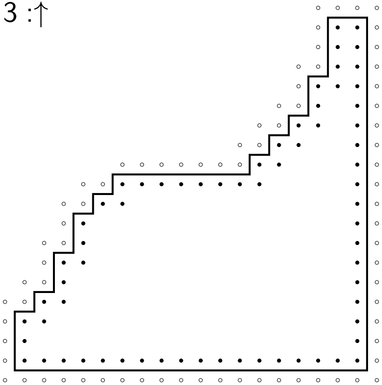
Adapted Version of Duval's Algorithm

Using the repetition properties of Christoffel words, we modify Duval's algorithm in order to stop the computation if the prefix read is not prefix of a Christoffel word.

Computation of the CMLP

$0 : \rightarrow,\ 1 : \downarrow,\ 2 : \leftarrow, 3 : \uparrow$

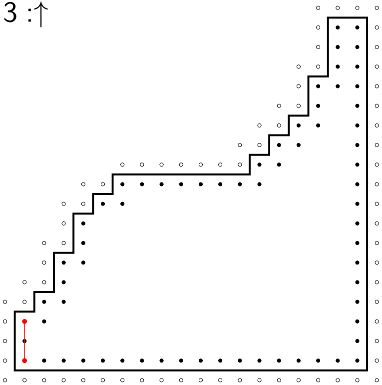
$3 < 0 < 1 < 2$



111010110110101000000010101011011100333333333333333333222222222222222222

$0 : \rightarrow,\ 1 : \downarrow,\ 2 : \leftarrow, 3 : \uparrow$

$3 < 0 < 1 < 2$

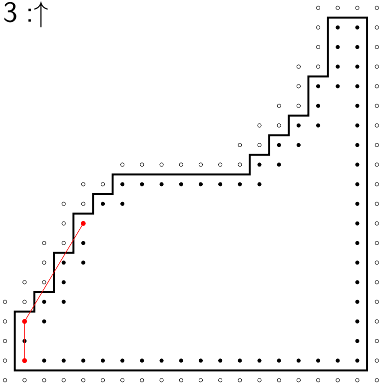


010110110101000000010101011011100333333333333333333222222222222222222

$(1)^3$

$0 : \rightarrow,\ 1 : \downarrow,\ 2 : \leftarrow, 3 : \uparrow$

$3 < 0 < 1 < 2$

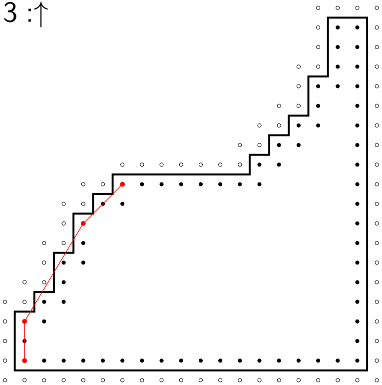


010100000001010101101110033333333333333333322222222222222222222

$(1)^3 \cdot (01011011)$

0 :$\rightarrow$, 1 :$\downarrow$, 2 :$\leftarrow$, 3 :$\uparrow$

$3 < 0 < 1 < 2$



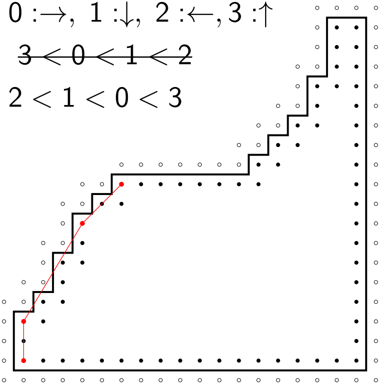
000000010101011011110033333333333333333322222222222222222

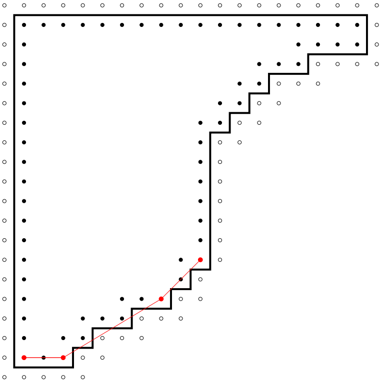
$(1)^3 \cdot (01011011) \cdot (0101)$

$0 :\rightarrow,\ 1 :\downarrow,\ 2 :\leftarrow, 3 :\uparrow$

~~$3 < 0 < 1 < 2$~~

$2 < 1 < 0 < 3$





000000010101011011100333333333333333333322222222222222222

$(1)^3 \cdot (01011011) \cdot (0101)$

Computation of the CMLP

$0 :\rightarrow,\ 1 :\downarrow,\ 2 :\leftarrow, 3 :\uparrow$

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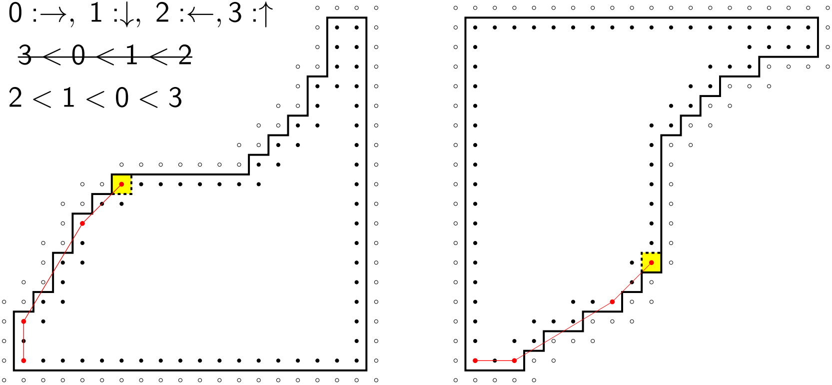


1000000101010110111003333333333333333332222222222222222222

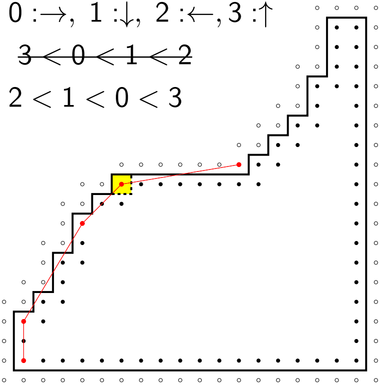
$(1)^3 \cdot (01011011) \cdot (0101)$

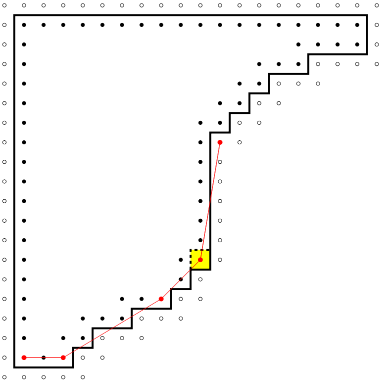
Computation of the CMLP

$0 : \rightarrow,\ 1 : \downarrow,\ 2 : \leftarrow, 3 : \uparrow$

~~$3 < 0 < 1 < 2$~~

$2 < 1 < 0 < 3$





10101011011100333333333333333333222222222222222222

$(1)^3 \cdot (01011011) \cdot (0101) \cdot (1000000)$

Computation of the CMLP

$0 :\rightarrow,\ 1 :\downarrow,\ 2 :\leftarrow, 3 :\uparrow$

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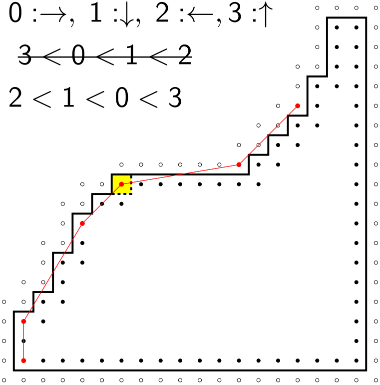

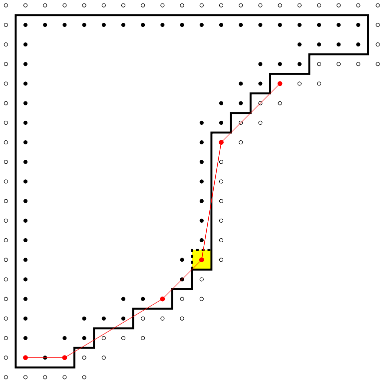

11011100333333333333333333222222222222222222

$(1)^3 \cdot (01011011) \cdot (0101) \cdot (1000000) \cdot (101010)$

Computation of the CMLP

0 :→, 1 :↓, 2 :←, 3 :↑

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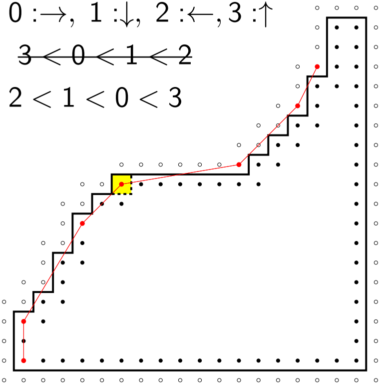

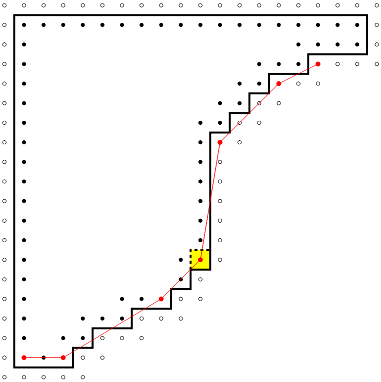

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(1)^3·(01011011)·(0101)·(1000000)·(101010)·(110)

$0 :\rightarrow,\ 1 :\downarrow,\ 2 :\leftarrow, 3 :\uparrow$

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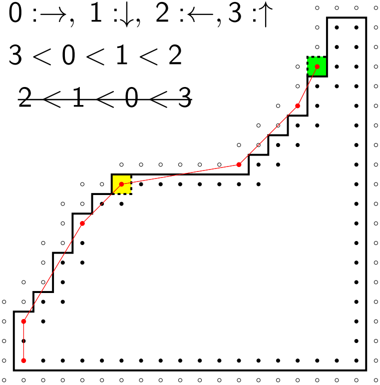

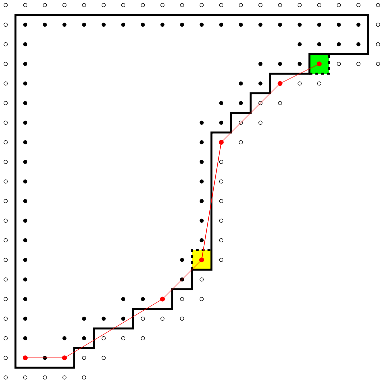

0110033333333333333333322222222222222222

$(1)^3 \cdot (01011011) \cdot (0101) \cdot (1000000) \cdot (101010) \cdot (110)$

Computation of the CMLP

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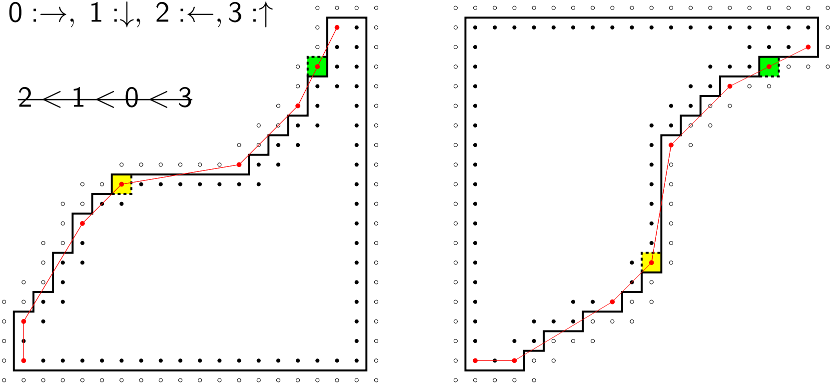

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$(1)^3 \cdot (01011011) \cdot (0101) \cdot (1000000) \cdot (101010) \cdot (110) \cdot (011)$

Both algorithms AMLP and CMLP have been implemented and are include in the ImaGene project.
http://gforge.liris.cnrs.fr/projects/imagene

Thank you!