Non-Convex Words

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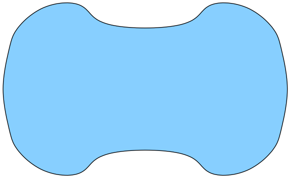
LIRMM

LAMA
Laboratoire de Mathématiques

Fonds de recherche sur la nature et les technologies
Québec

Discrete Geometry

Continuous geometrical object $\longrightarrow$ Discretization process $\longrightarrow$ Discrete image



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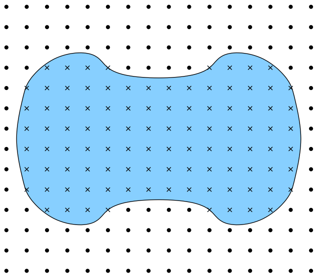

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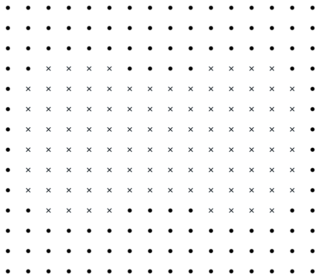
Continuous geometrical object ⟶ Discretization process ⟶ Discrete image



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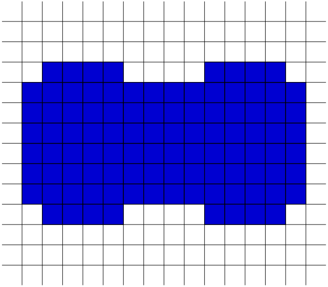

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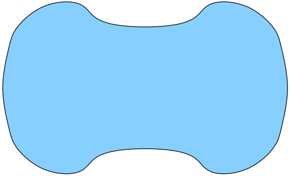

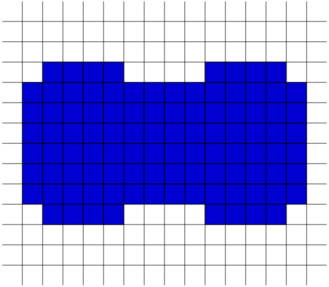

Digital Convexity

Definition

A *polyomino* is the interior of a closed non-intersecting grid path.

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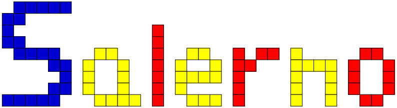

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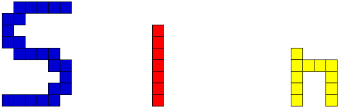

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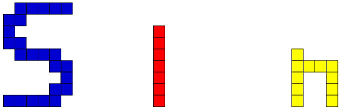

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A polyomino is *hv*-convex if all of its columns and all of its rows are connected.

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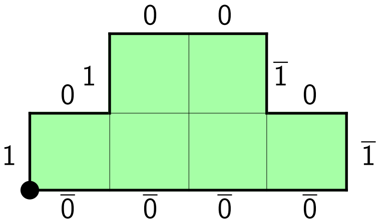
Definition

A polyomino is *digitally convex* if it corresponds to the discretization of some convex set.

Freeman chain code

$$\Sigma = \{0, \overline{0}, 1, \overline{1}\}$$

$0 \rightarrow$	$1 \uparrow$
$\overline{0} \leftarrow$	$\overline{1} \downarrow$



$$w = 1\,0\,1\,0\,0\,\overline{1}\,0\,\overline{1}\,\overline{0}\,\overline{0}\,\overline{0}\,\overline{0}.$$

Notation

The word w is a *boundary word* of the polyomino P.

Quadrant words

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Given w a boundary word of the hv-convex polyomino P, a non-extendable factor of ww over two letters is called a *quadrant* word.

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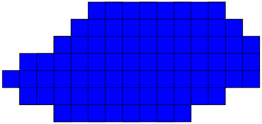
All hv-convex polyominoes admits four quadrant words :



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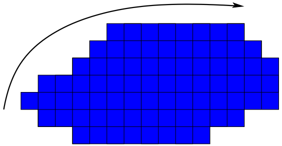

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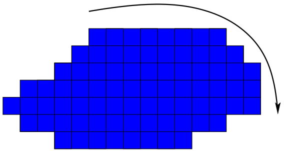

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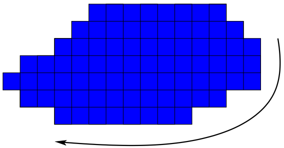

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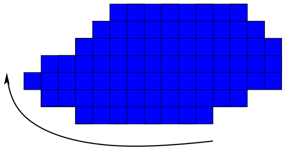

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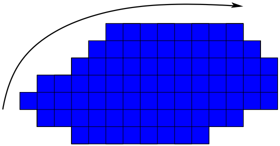


Lyndon Words

Definition

A word w over an ordered alphabet is a Lyndon word if for all non-empty words u and v :

$$w = uv \implies w < vu.$$

(where $<$ denotes the lexicographic order.)

Theorem (Lyndon, 1950)

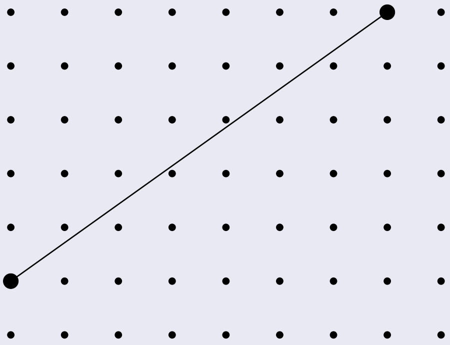
Any word w over an ordered alphabet admits a unique factorization as decreasing Lyndon words.

Notation

Given a word w, its unique factorization as decreasing Lyndon words $w = l_1 \cdot l_2 \cdots l_m$ is noted : $w = (l_1, l_2, \ldots, l_m)_{\textbf{Lyn}}$.

Christoffel Words

Definition (Borel et Laubie, 1993)



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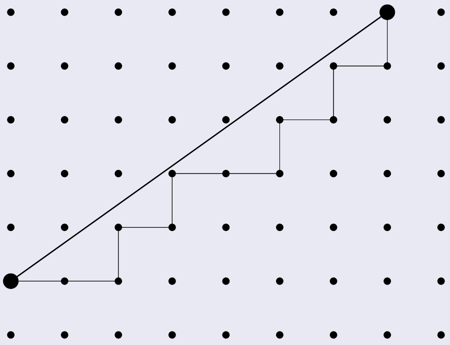

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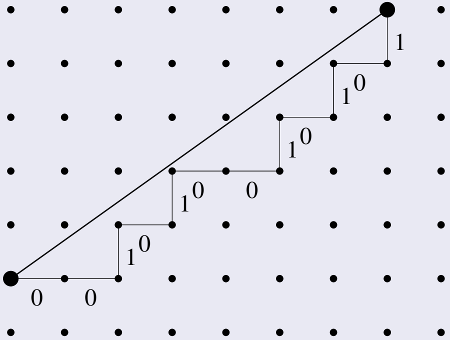

$$\mathcal{C}_{5/7} = 001010010101.$$

Some Properties of Christoffel Words

Definition

A word $w \in \mathcal{A}^*$ is *balanced* if for all $a \in \mathcal{A}$ and all pair of factors u, v of w,

$$|u| = |v| \implies ||u|_a - |v|_a| \leq 1.$$

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Theorem (Berstel, de Luca, 1997)

Over an ordered two letter alphabet, the set of Christoffel words is exactly the set of balanced Lyndon words.

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Each non-trivial Christoffel word w has a unique factorization as two Christoffel $w = uv$.

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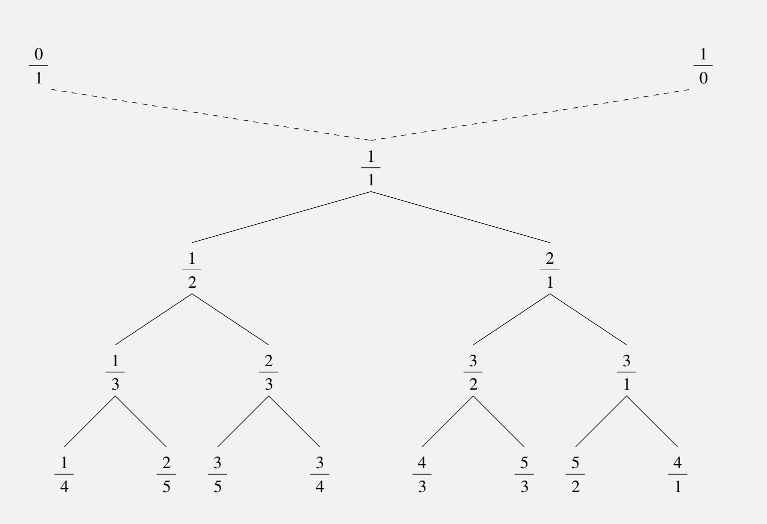

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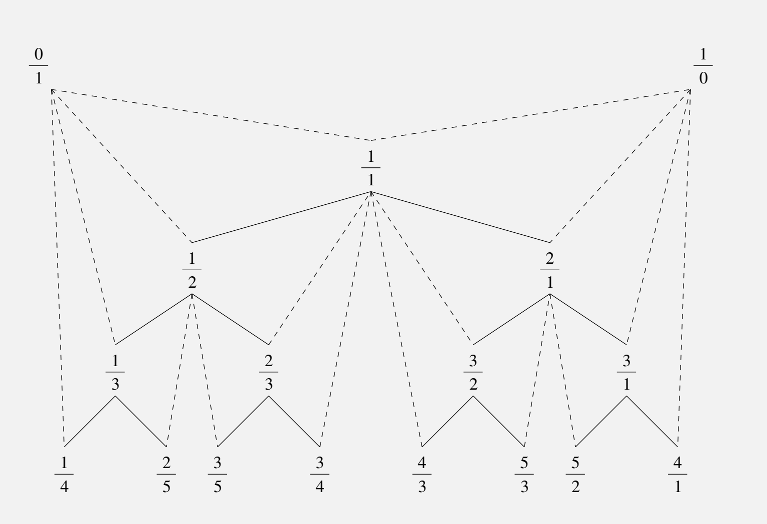

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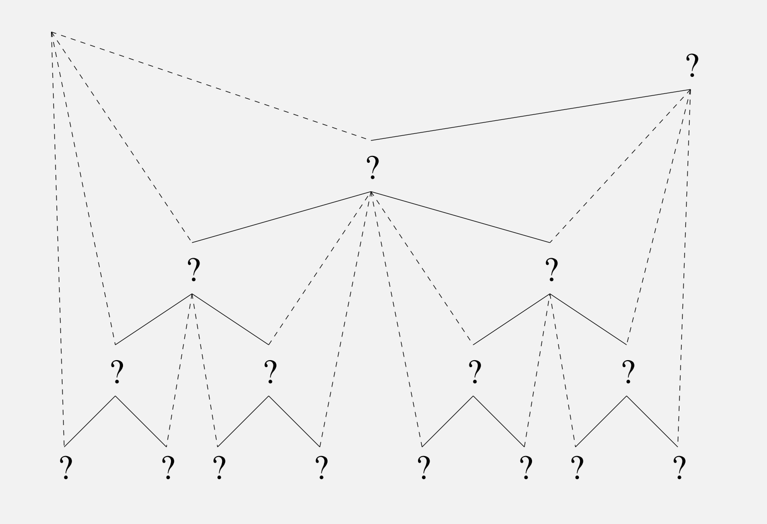

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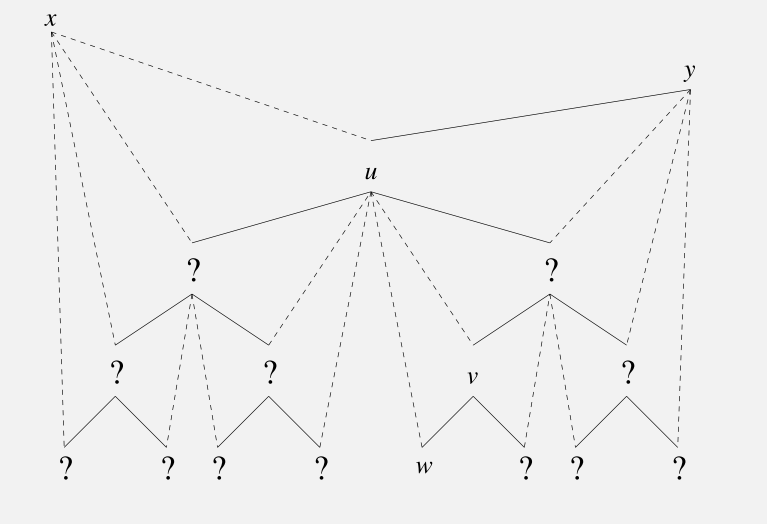

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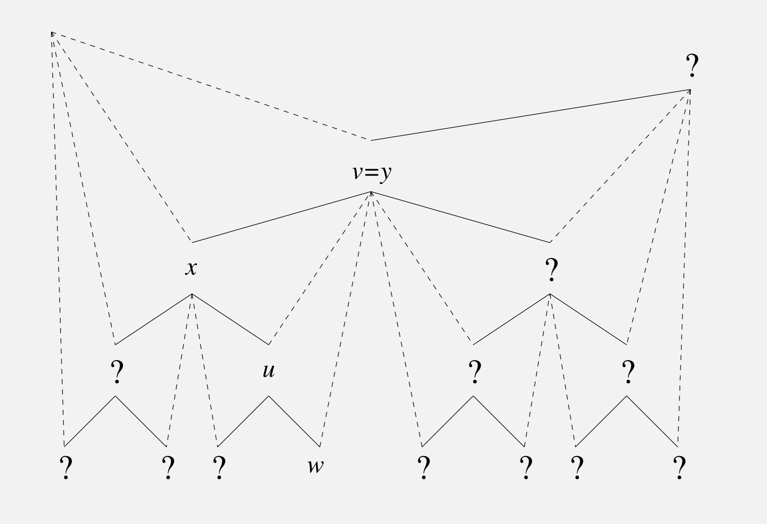

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- $v = (x, y)_{\mathbf{C}} \implies x \leq u < w < v < y$.

Left and Right Factorization

Consider the Christoffel word $\mathcal{C}_{5/8} = 0010010100101$.

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$$0010010100101$$
$$00100101 \cdot 00101$$

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$$\begin{array}{r} 0010010100101 \\ 00100101 \cdot 00101 \\ 001 \cdot 00101 \cdot 00101 \end{array}$$

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\begin{array}{r}
0010010100101 \\
00100101 \cdot 00101 \\
001 \cdot 00101 \cdot 00101 \\
0 \cdot 01 \cdot 00101 \cdot 00101
\end{array}
$$

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$$\begin{array}{rcl} 0010010100101 & & 0010010100101 \\ 00100101 \cdot 00101 & & \\ 001 \cdot 00101 \cdot 00101 & & \\ 0 \cdot 01 \cdot 00101 \cdot 00101 & & \\ (0, 01, 00101, 00101)_{Left} & & \end{array}$$

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$$
\begin{array}{r}
0010010100101 \\
00100101 \cdot 00101 \\
001 \cdot 00101 \cdot 00101 \\
0 \cdot 01 \cdot 00101 \cdot 00101 \\
(0, 01, 00101, 00101)_{Left}
\end{array}
\qquad
\begin{array}{l}
0010010100101 \\
00100101 \cdot 00101 \\
00100101 \cdot 001 \cdot 01
\end{array}
$$

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Proposition

Given a Chritoffel word $w = 0x1$ with the factorizations

$$w = (0, l_1, l_2, \ldots, l_m)_{Left} = (r_1, r_2, \ldots, r_n, 1)_{Right}$$

then

$$x1 = (l_1, l_2, \ldots, l_m)_{\mathbf{Lyn}} \text{ and } 0x = (r_1, r_2, \ldots, r_n)_{\mathbf{Lyn}}$$

Characterization of Digital Convexity

Theorem (Brlek, Lachaud, Provençal, Reutenauer, 2009)

A polyomino is digitally convex if and only if each one of its quadrant words w is such that $w = (l_1, l_2, \ldots, l_m)_{\mathbf{Lyn}}$ where each l_i is a Christoffel words.

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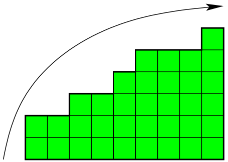
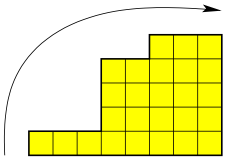


1 1 0 0 1 0 0 1 0 1 0 0 0 1 0



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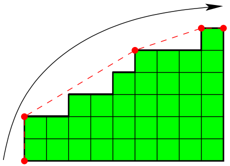

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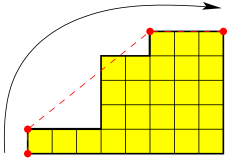
$(1, 1, 00100101, 0001, 0)_{\mathbf{Lyn}}$



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A polyomino is digitally convex if and only if each one of its quadrant words w is such that $w = (l_1, l_2, \ldots, l_m)_{\mathbf{Lyn}}$ *where each* l_i *is a Christoffel words.*

Remark

An optimal algorithm to test digital convexity is deduced from this theorem.

Convex Words

Definition

Let **CV** be the set of all convex words on $\mathcal{A}$, that is the set

$$\mathbf{CV} = \{(l_1, l_2, \ldots, l_m)_{\mathbf{Lyn}} \in \mathcal{A}^* \mid \text{all } l_i \text{ are Christoffel words}\}.$$

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Property (Reutenauer, 2008)

The language **CV** *is factorial.*

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Let **NC** be the set of all non-convex words on $\mathcal{A}$, that is $\mathbf{NC} = \mathcal{A}^* \setminus \mathbf{CV}$.

Property

The language **NC** *forms a monoidal ideal of* $\mathcal{A}^*$.

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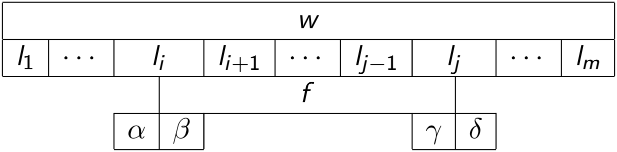

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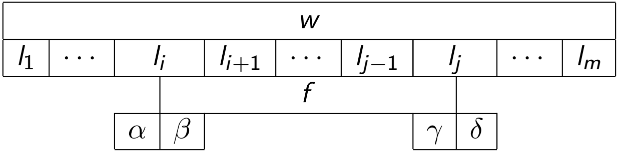

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Convex Words

Definition

Let **CV** be the set of all convex words on $\mathcal{A}$, that is the set

$$\mathbf{CV} = \{(l_1, l_2, \ldots, l_m)_{\mathbf{Lyn}} \in \mathcal{A}^* \mid \text{all } l_i \text{ are Christoffel words}\}.$$

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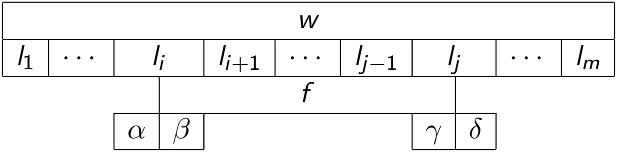

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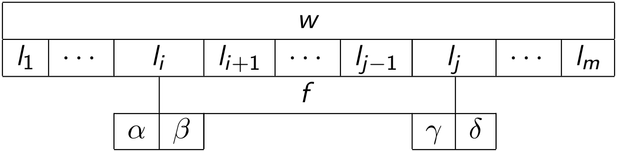

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Non-Convex Minimal

Definition

Let **NCM** be the set :

$$\mathbf{NCM} = \{w \in \mathbf{NC} \,|\, \forall x \in \mathrm{Factor}(w) \setminus \{w\}, x \in \mathbf{CV}\}.$$

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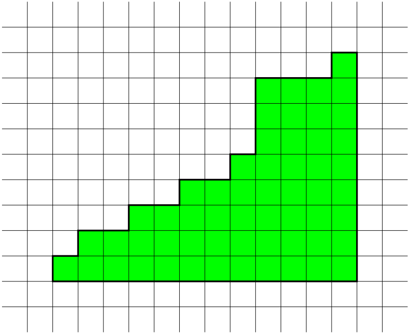
Proposition

The language **NC** *is a monoidal ideal of* $\mathcal{A}^*$ *generated by* **NCM**.

Example



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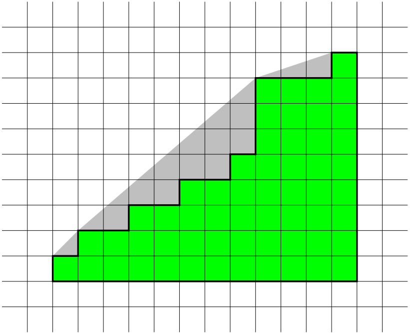

$$10100100100101110001 0 \in \mathbf{NC}$$

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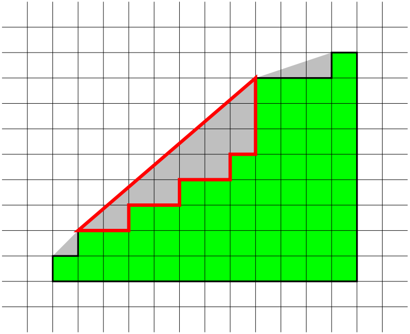

$0010010010111 \in \mathbf{NC}$

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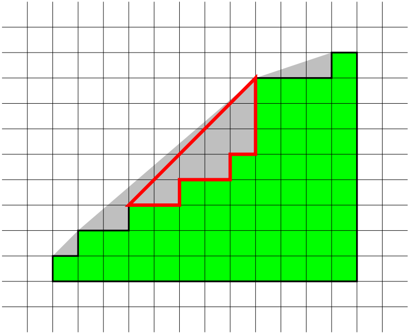

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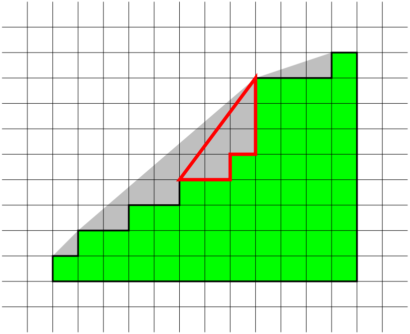

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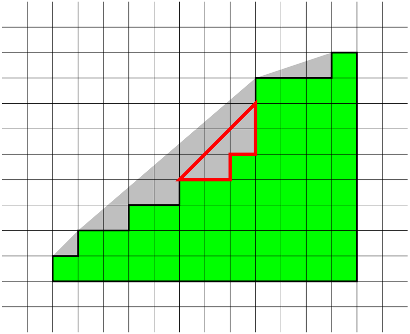
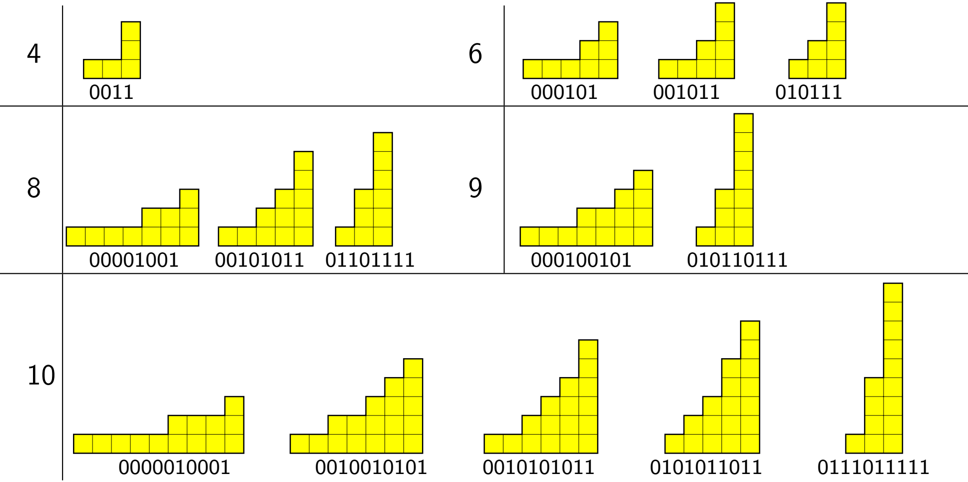


$$001011 \in \mathbf{NCM}$$

Shortest Elements of **NCM**



Main Result

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The set $\mathbf{NCM} = \left\{ uw^k v \mid w = (u,v)_{\mathbf{C}} \text{ and } k \geq 1 \right\}$.

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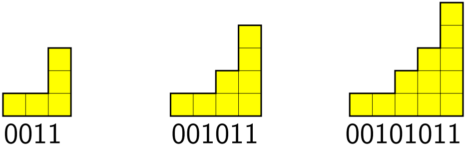

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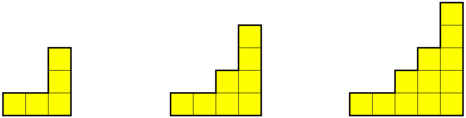

0011 — $0 \cdot (01) \cdot 1$

001011 — $0 \cdot (01)^2 \cdot 1$

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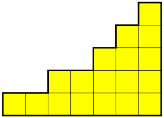

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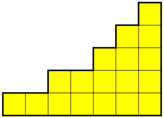

0010010101

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$u, v, uw^k, w^k v$ are Christoffel words so

- $0X1$ is a Lyndon word since it is the concatenation of increasing Lyndon words.
- $0X1$ is not a Christoffel word since it admits two factorizations as Christoffel words.

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- $X1 \in \mathbf{CV}$ is shown in a similar way.

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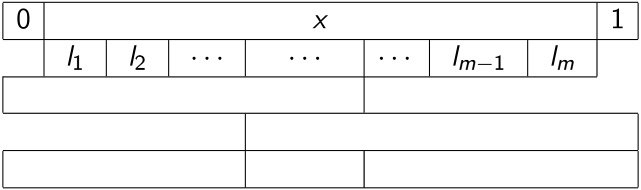

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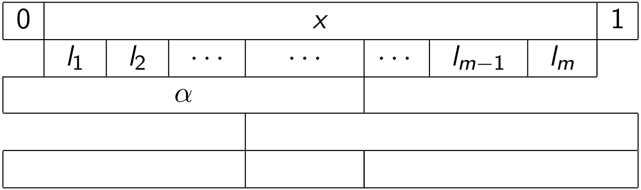

$$0 < l_1 \geq l_2 \geq l_3 \geq \cdots$$

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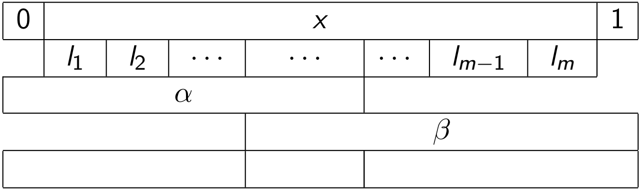

$$\cdots \geq l_{m-1} \geq l_m < 1$$

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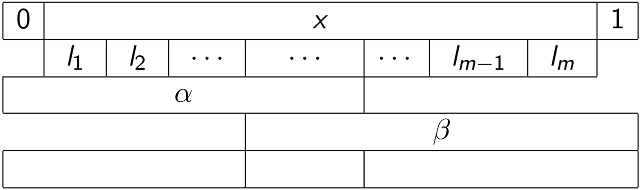

$0x \in \mathbf{CV}$ so α is a Christoffel word.
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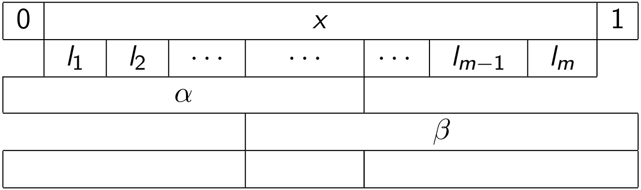

$\alpha = (0, l_1, l_2, \ldots)_{Left}$ so for all l_i that is in α

$$\Longrightarrow \; 0l_1 \cdots l_i = (0l_1 \cdots l_{i-1}, l_i)_{\mathbf{C}}$$

$$\Longrightarrow \; l_{i-1} = l_i \text{ or } |l_{i-1}| < |l_i|.$$

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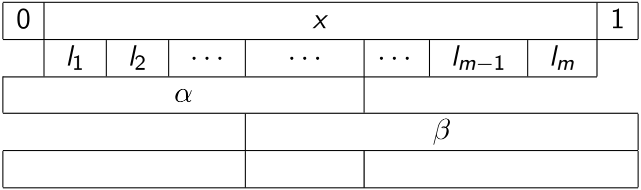

$$\beta = (\ldots, l_{m-1}, l_m, 1)_{Right} \text{ so for all } l_i \text{ that is in } \beta$$

$$\implies l_i l_{i+1} \cdots l_m 1 = (l_i, l_{i+1} \cdots l_m 1)_{\mathbf{C}}$$

$$\implies l_i = l_{i+1} \text{ or } |l_i| > |l_{i+1}|$$

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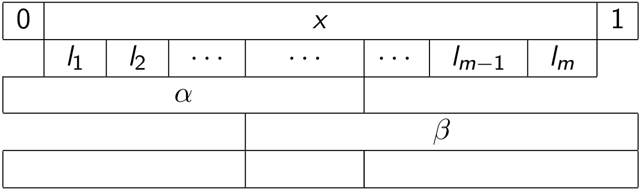

if l_i and l_{i+1} are in both α and β

$$l_i = l_{i+1}$$

let us call them w.

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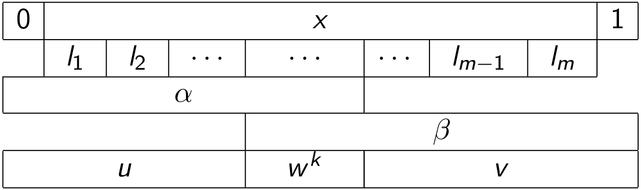

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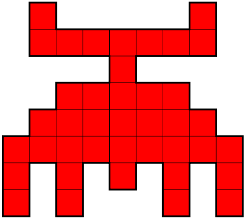

Let w be a contour word, if w is not digitally convex, it must admit as a factor at least one word of **NCM** or a word of the form $ab^k\overline{a}$ where $(a,b) \in \{(0,1),(\overline{1},0),(\overline{0},\overline{1}),(1,\overline{0})\}$.

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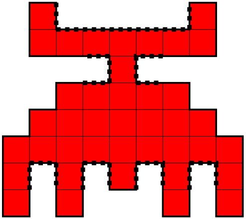

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Let $z = uw^k v \in \mathbf{NCM}$, then :

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- For all $x, y \in \mathrm{Factor}(z)$,

$$|x| = |y| \implies ||x|_a - |y|_a| = 1,$$

Except for the cases where x is a prefix of z, y is a suffix of z and $|x| = |y| = n|w|$, where $n \in \{1, 2, \ldots, k\}$. In these cases $||x|_a - |y|_a| = 2$.

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Since $\max\{|w| : w \in \mathbf{NCM}\} = \infty$, it is not the case in discrete geometry.

GRAZIE!