A sub-quatratic algorithm to determine if a polyomino tiles the plane by translation

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LIRMM (Montpellier),
LAMA (Chambéry)

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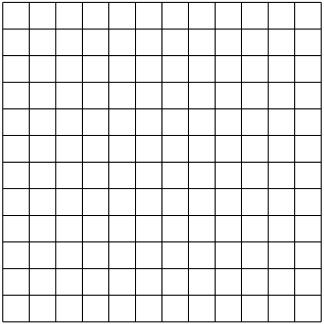
Introduction to polyominos

- Discrete plane : $\mathbb{Z}^2$



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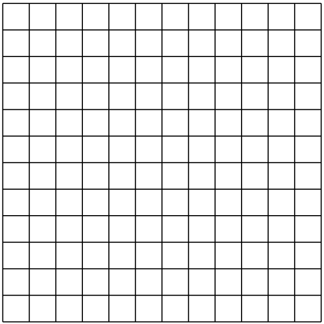

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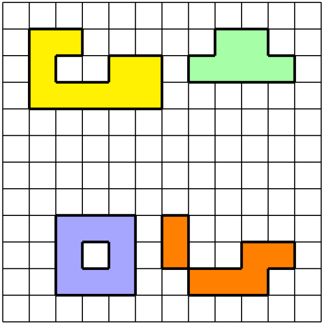

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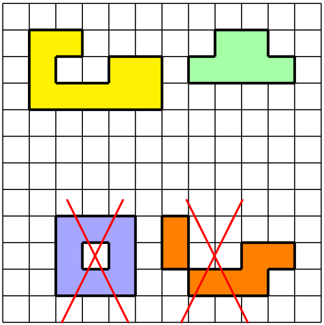

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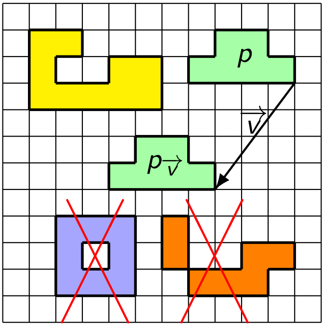
- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A *polyomino* is a finite, 4-connected subset of the plane, without holes.
- **Notation** : Let p be a polyomino and $\vec{v}$ a vector of $\mathbb{Z}^2$, $p_{\vec{v}}$ will denote the image of p by de translation $\vec{v}$.



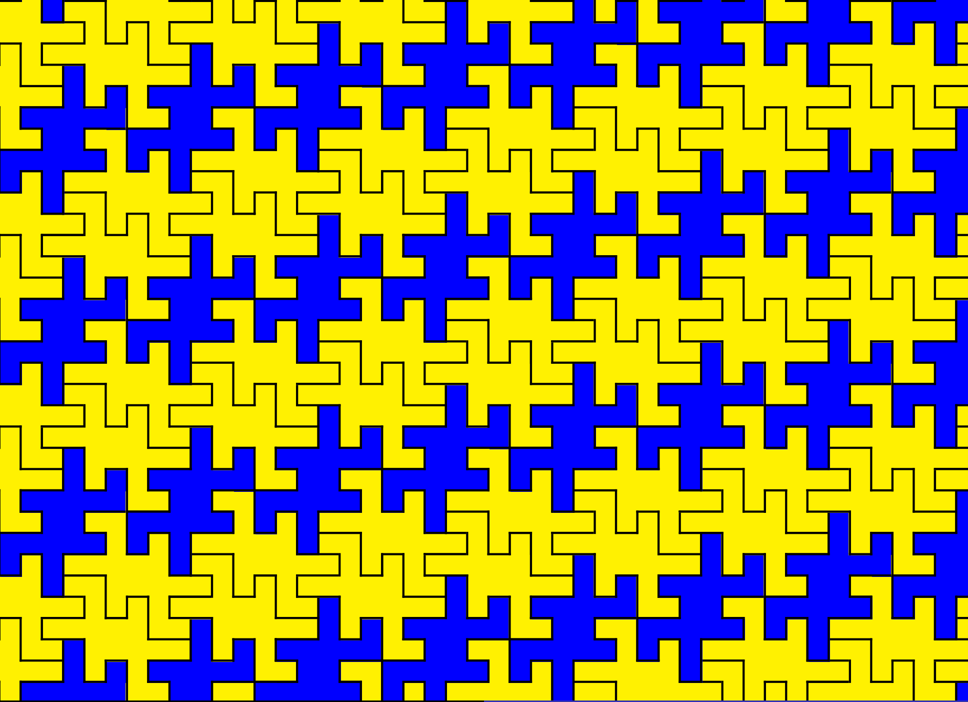
Tiling

Definition

A tiling of the discret plane $\mathbb{Z}^2$ *by a set of polyominos* $\mathcal{P}$ *is a subset* $\mathcal{T} \subset \mathcal{P} \times \mathbb{Z}^2$ *such that :*

1. $\displaystyle\bigcup_{(P,\overrightarrow{v})\in\mathcal{T}} P_{\overrightarrow{v}} = \mathbb{Z}^2$
2. *For any distinct pair* $(P, \overrightarrow{u}), (Q, \overrightarrow{v}) \in \mathcal{T}$, $P_{\overrightarrow{u}} \cap Q_{\overrightarrow{v}} = \emptyset$.

Example



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Theorem (Ollinger, 2008)

The tiling problem with $|\mathcal{P}| = 11$ is undecidable.

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We say that a polyomino P tiles the plane if there exist a tiling of the plane by $\{P\}$.

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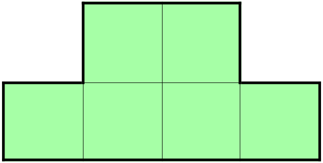
Theorem (Gambini and Vuillon, 2003)

There exist an algorithm in $\mathcal{O}(n^2)$ time that tests if a polyomino tiles the plane.

The Beauquier-Nivat characterization

Freeman chain code

$$\Sigma = \{0, \overline{0}, 1, \overline{1}\}$$



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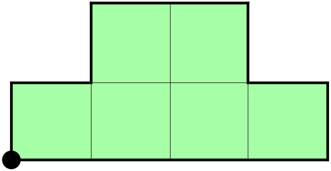

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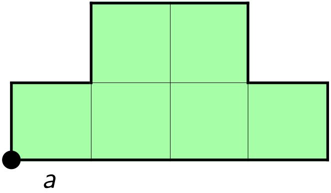

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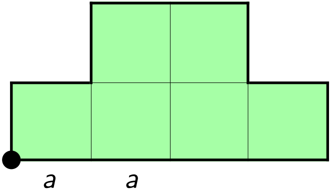

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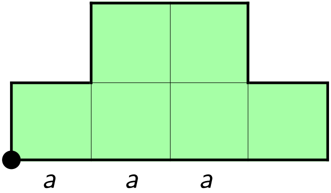

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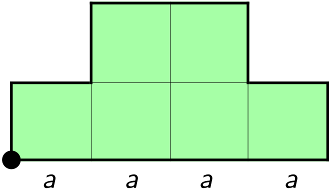

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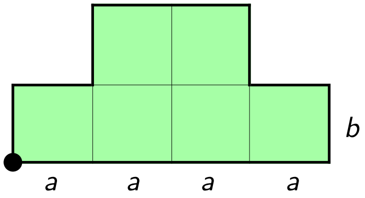

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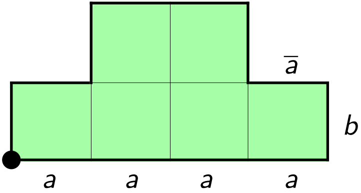

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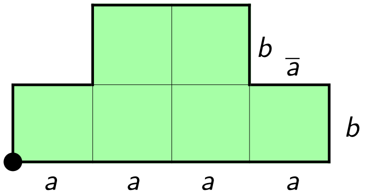

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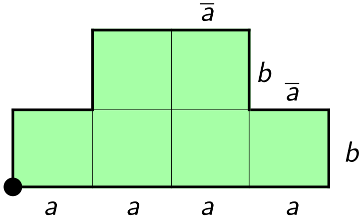

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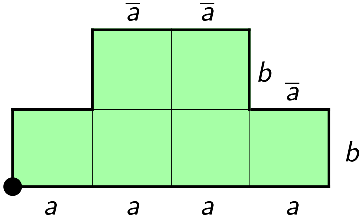

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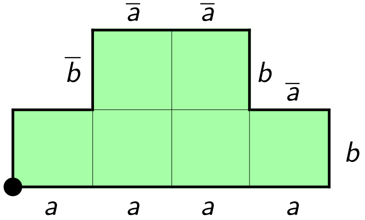

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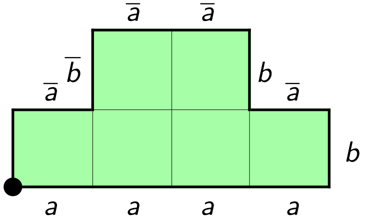

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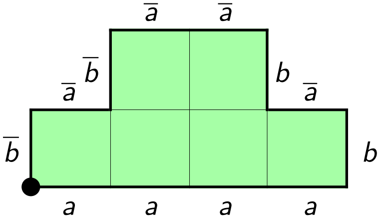

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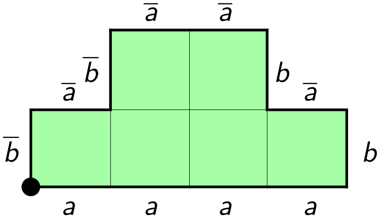

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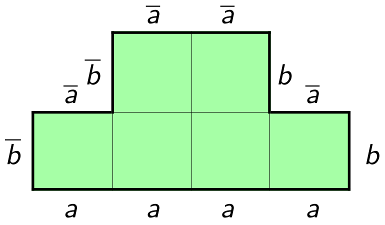

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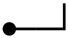

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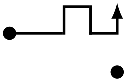

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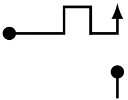

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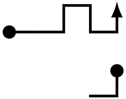

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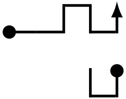

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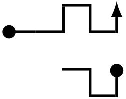

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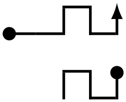

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Let $u, v, w \in \Sigma^* = \{a, \bar{a}, b, \bar{b}\}^*$ such that $w = uv$,
$\widehat{w} = \widehat{uv} = \widehat{v}\widehat{u}$ and $w = \widehat{\widehat{w}}$.

$u = a\ a\ b\ a\ \bar{b}\ a\ b$ 

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Characterization

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Theorem (Beauquier and Nivat, 1991)

A polyomino P tiles the plane if and only its boundary word $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$ for some $X, Y, Z \in \Sigma^*$.

Neighbouring

Definition

Two polyominoes p and q are simply neighbouring if

- *They are adjacent.*
- *They don't overlap.*
- *They don't form a hole.*

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Three polyominoes p, q and r form a triad if

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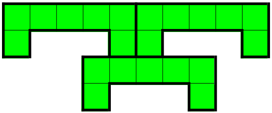

Surrounding

Definition

A surrounding of the polyomino p is an ordered sequence of translated copies $(p_0, p_1, \ldots, p_{k-1})$ such that for every i from 0 to $k-1$, the polyominoes p, p_i and p_{i+1} form a triad.

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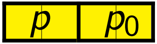

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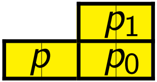

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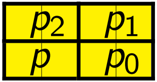

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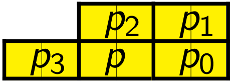

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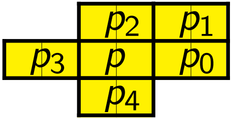

Surroundings and tilings

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If the polyominoes p and q are simply neighbouring then for any vector $\overrightarrow{u}$ the polyominoes $p_{\overrightarrow{u}}$ and $q_{\overrightarrow{u}}$ are also simply neighbouring.

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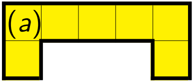
Proposition

A polyomino p tiles the plane if and only if it admits a surrounding.

Surroundings and tilings



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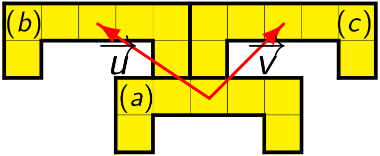

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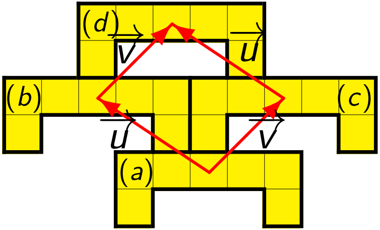

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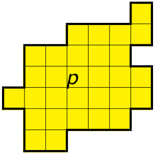

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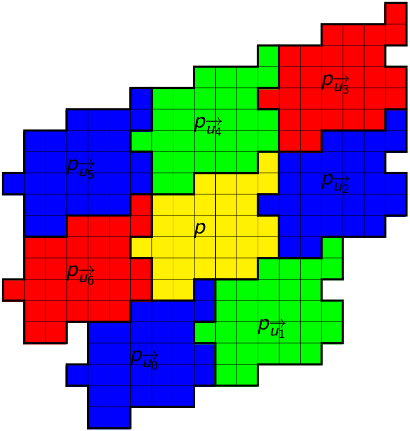

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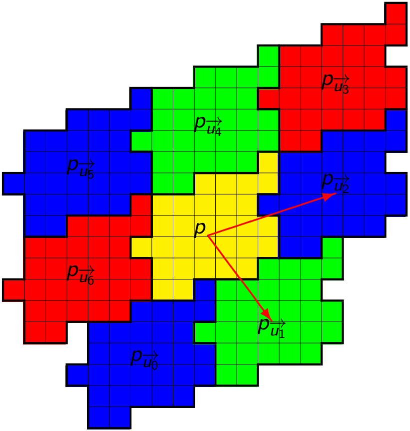

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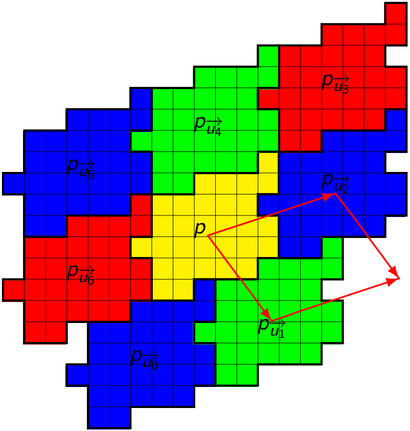

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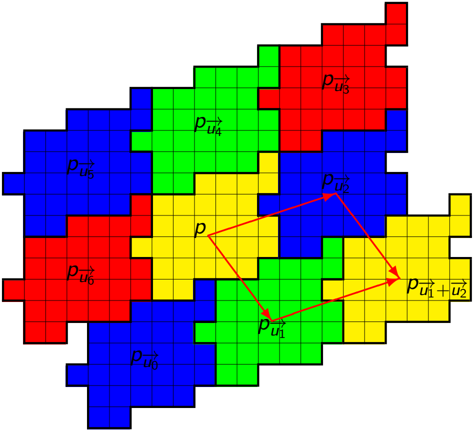

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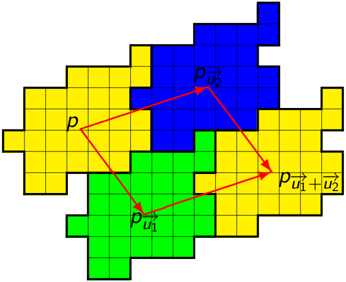

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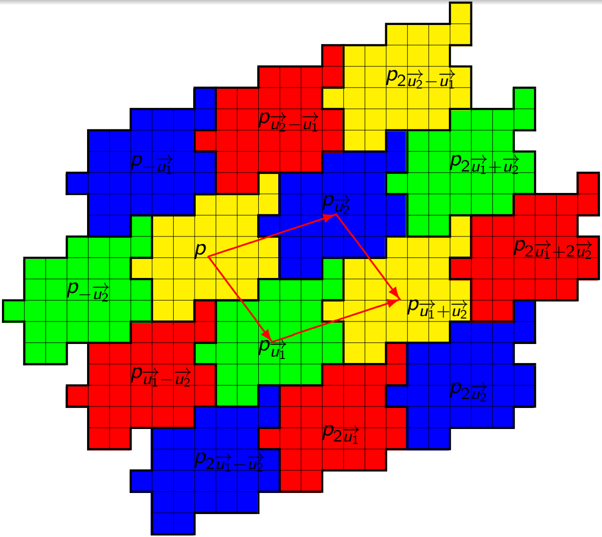

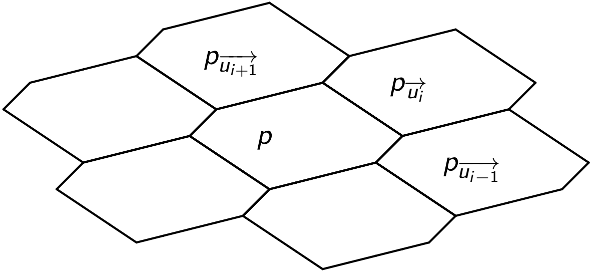
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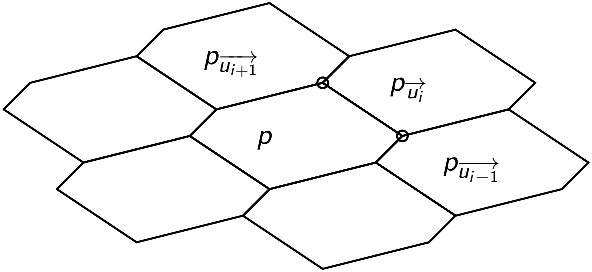
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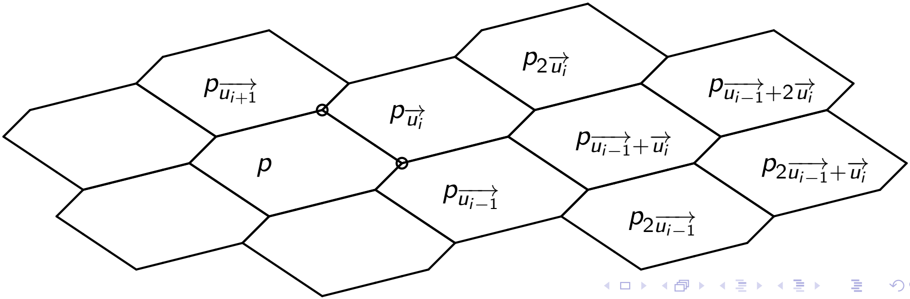
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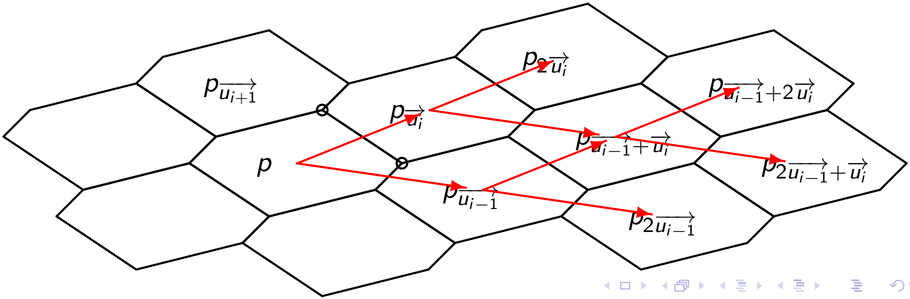
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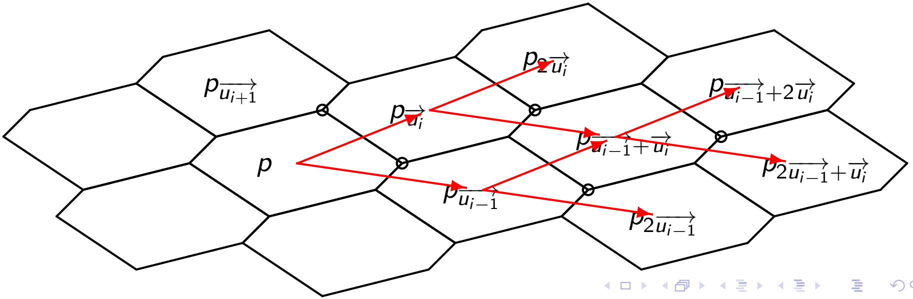
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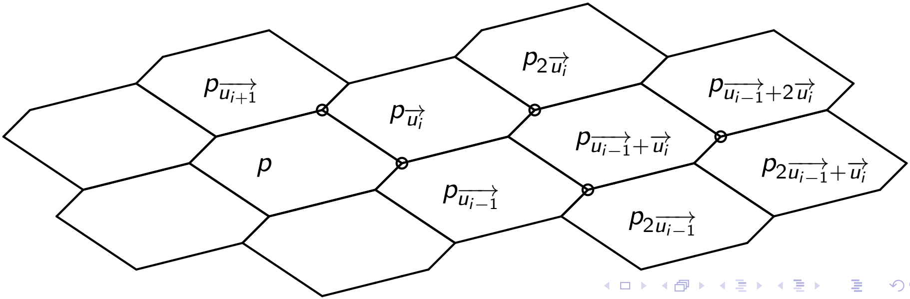
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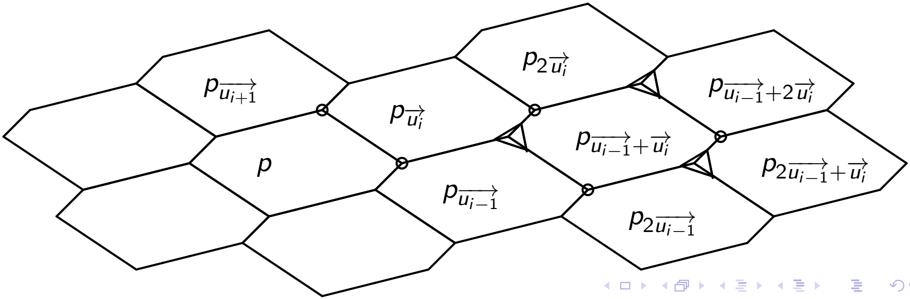
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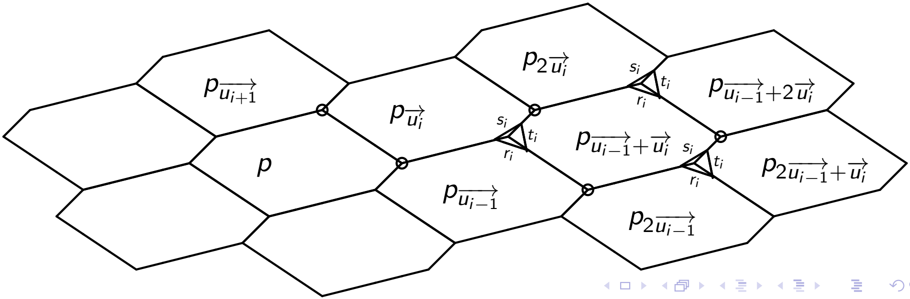
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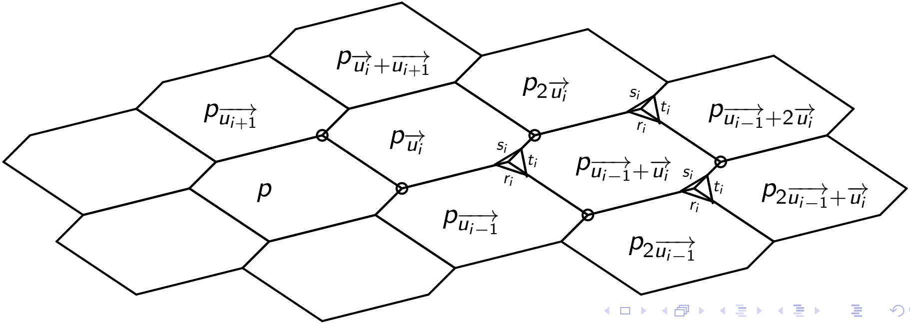
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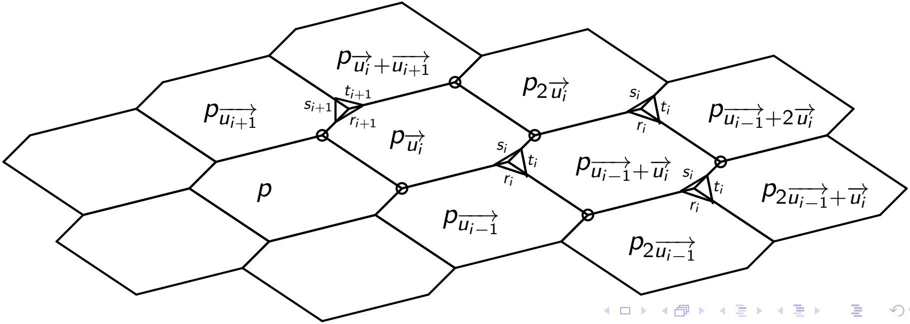
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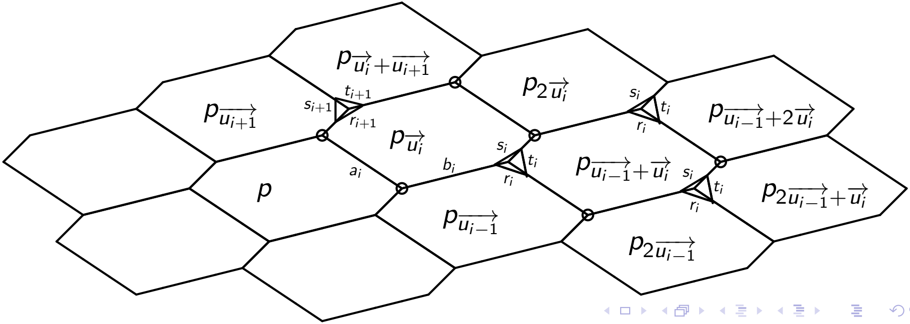
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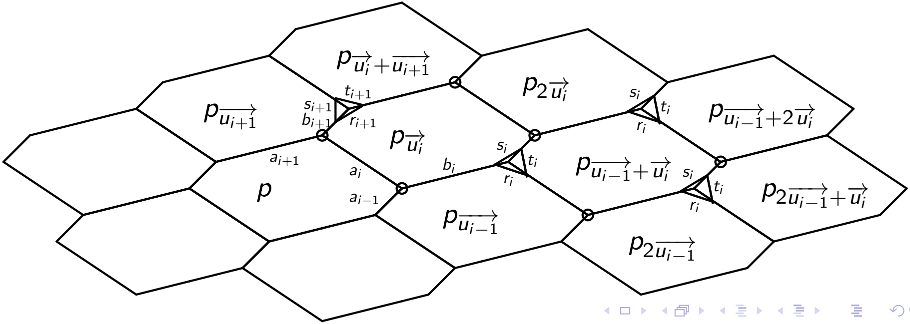
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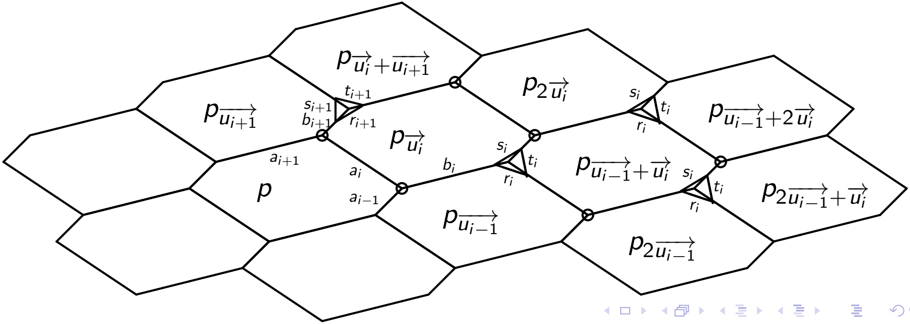

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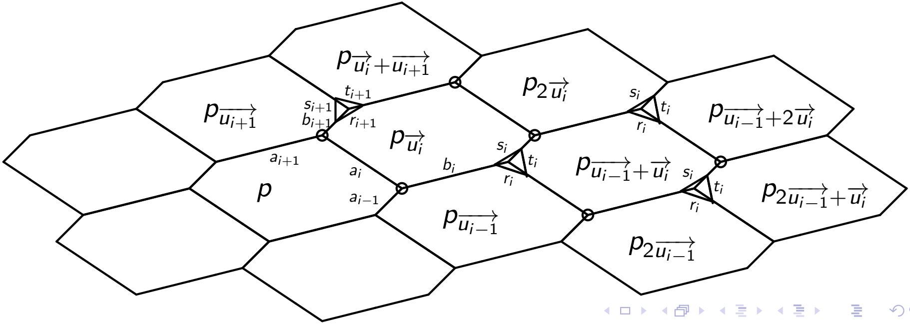

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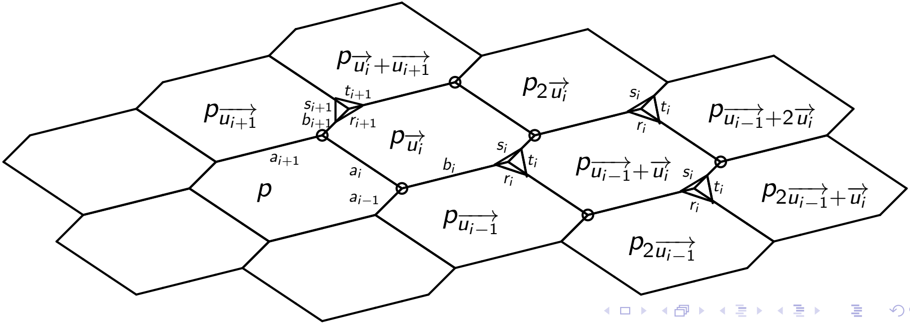

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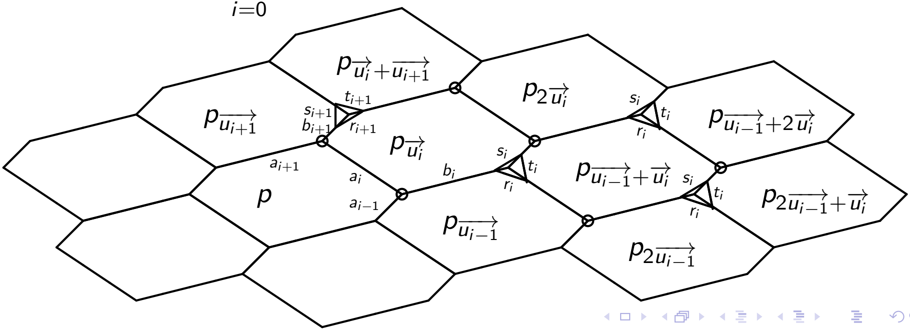

Surroundings and the factorization

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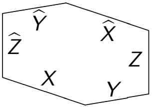

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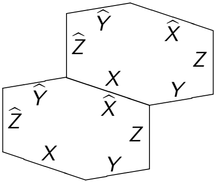

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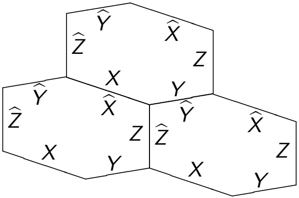

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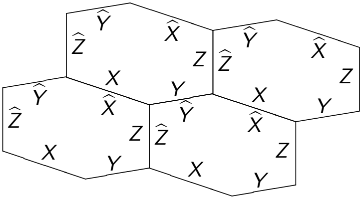

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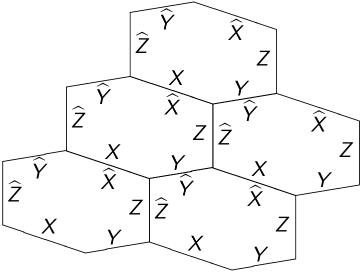

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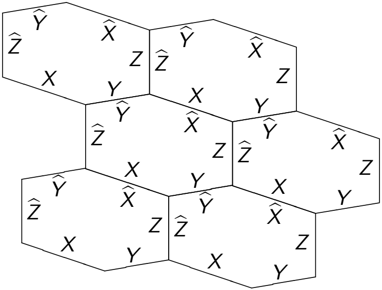

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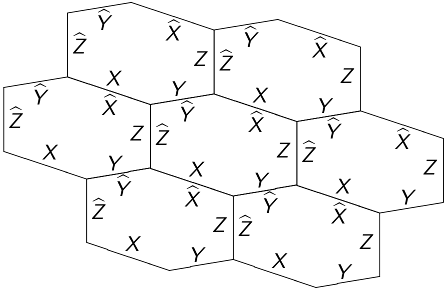

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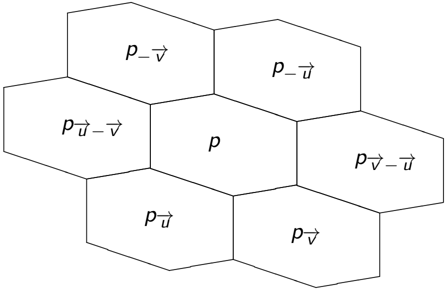

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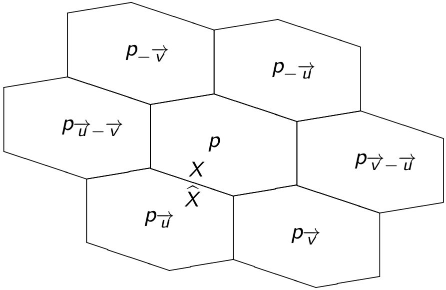

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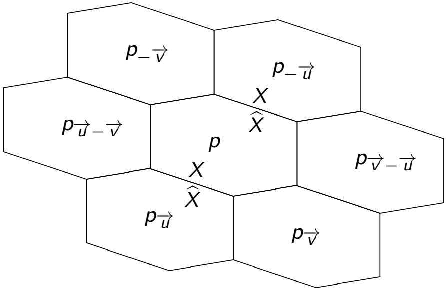

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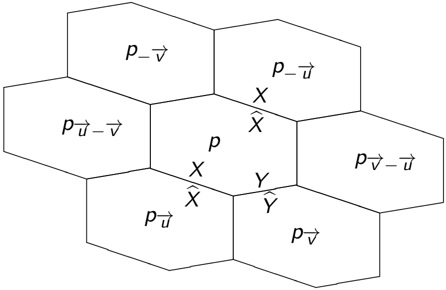

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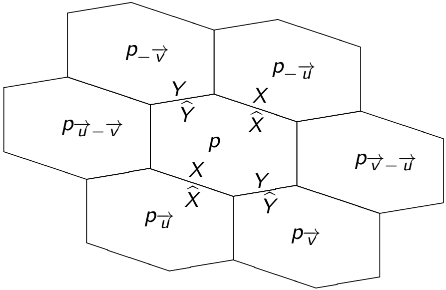

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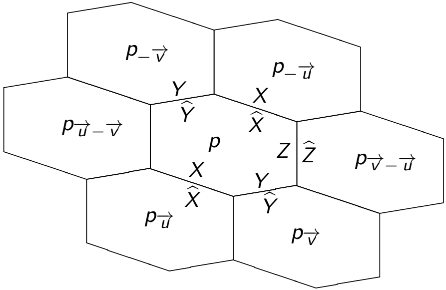

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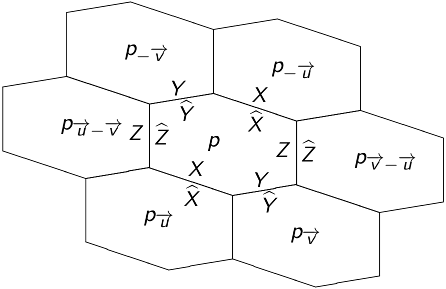

Pseudo-square and pseudo-hexagons

Definition

A polyomino with Beauquier-Nivat factorization $XYZ\widehat{X}\widehat{Y}\widehat{Z}$ *is called a pseudo-square if one of the factors* X, Y, Z *is the empty word. It is called a pseudo-hexagon otherwise.*

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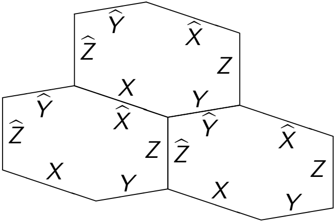
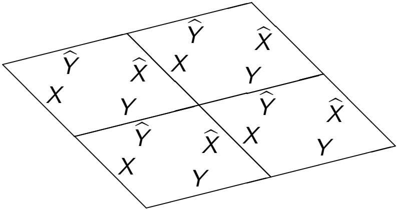


Pseudo-square

$w \equiv XY\widehat{X}\widehat{Y}.$



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Given an occurence of a factor A in the word w coding a polyomino p. This occurence of the factor A is admissible if

- $w \equiv Ax\widehat{A}y$, *for* x, y *such that* $|x| = |y|$.
- A *is maximal, that is,* $\mathrm{first}(x) \neq \overline{\mathrm{last}(x)}$ *and* $\mathrm{first}(y) \neq \overline{\mathrm{last}(y)}$.

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Notation

Given an admissible occurence of a factor A in w, the occurence of the factor $\widehat{A}$ such that $w \equiv Ax\widehat{A}y$ as in the previous definition is called its homologue.

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Let $\mathcal{A}$ be the set of all admissible factors overlapping a position α in w and $\widehat{\mathcal{A}}$ be the set of their respective homologous factors. Then, there is at least one position in w that is not covered by any element of $\mathcal{A} \cup \widehat{\mathcal{A}}$.

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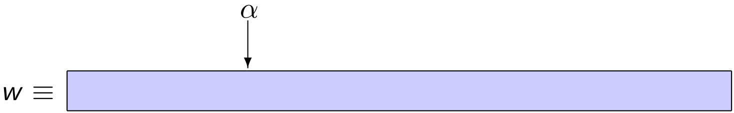

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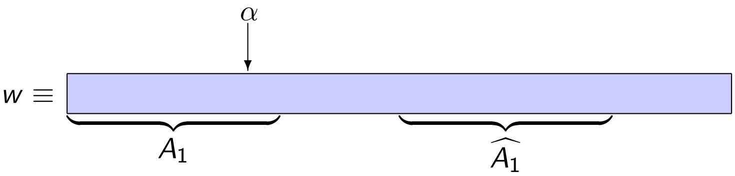

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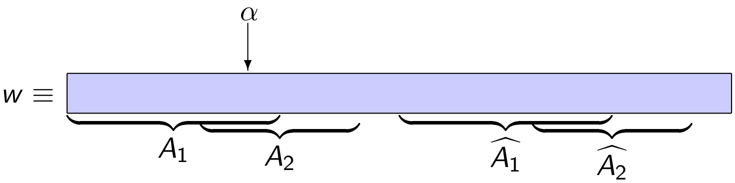

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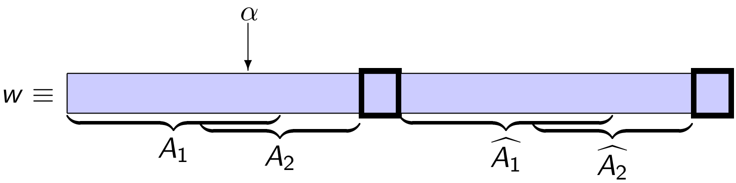

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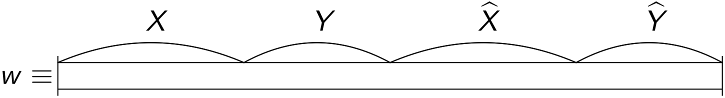

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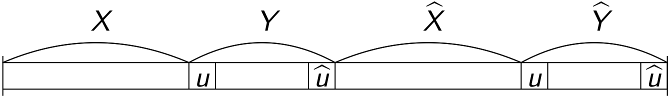
$w \equiv$ 

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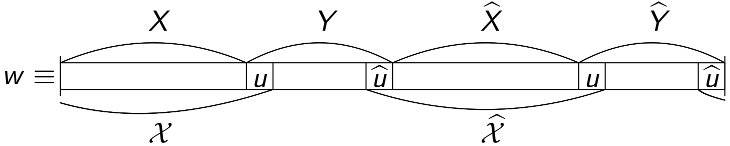

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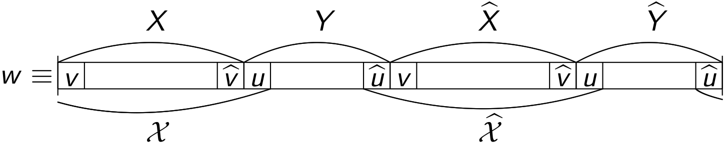

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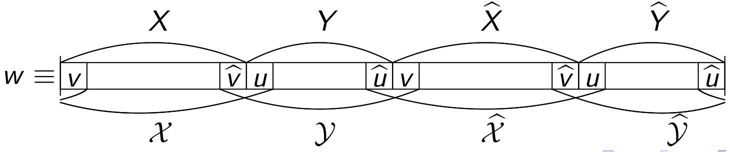

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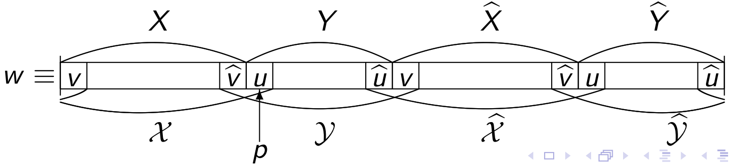

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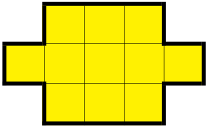

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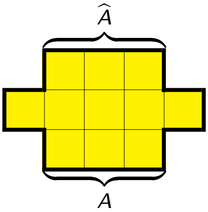

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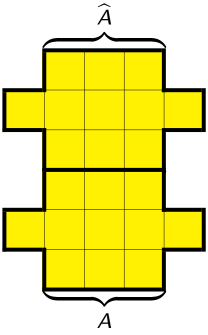

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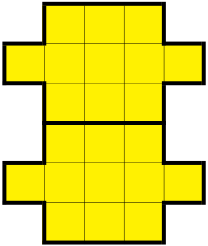

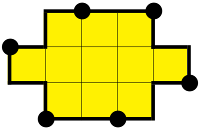

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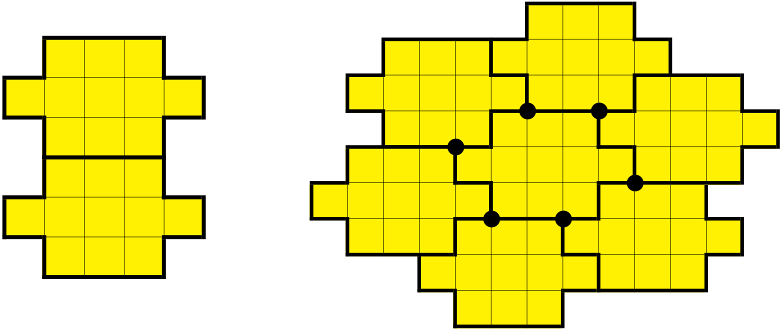

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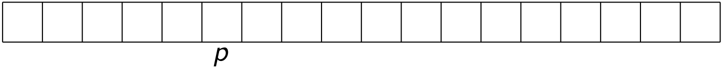

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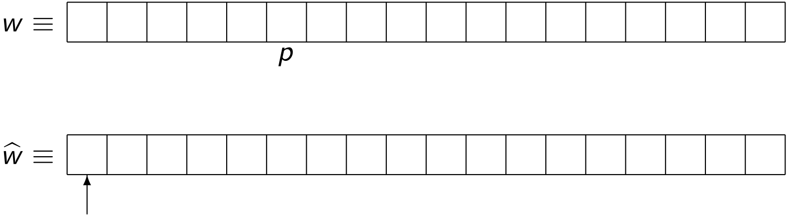

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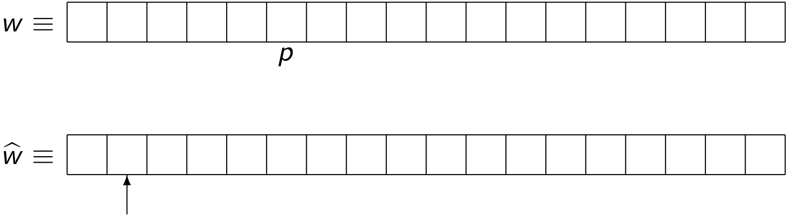

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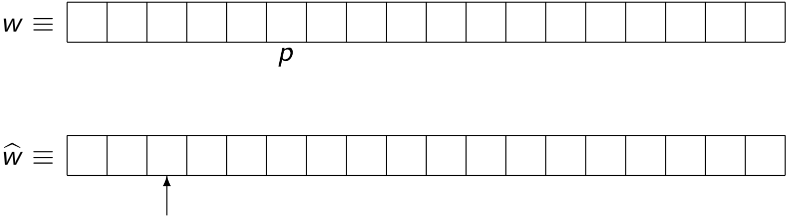

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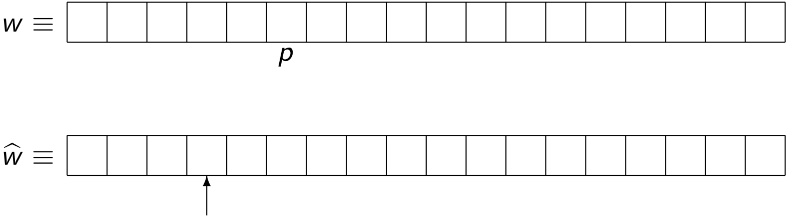

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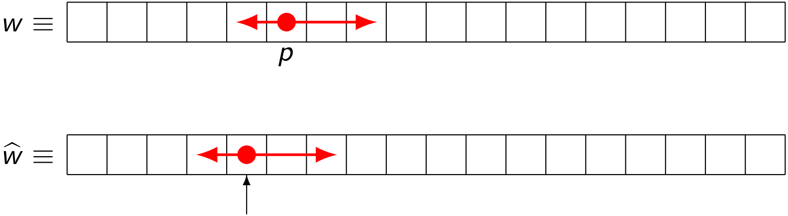

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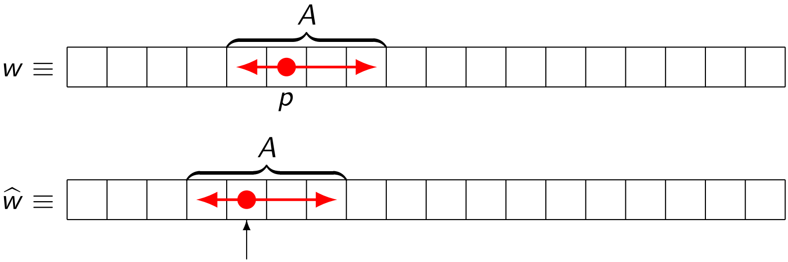

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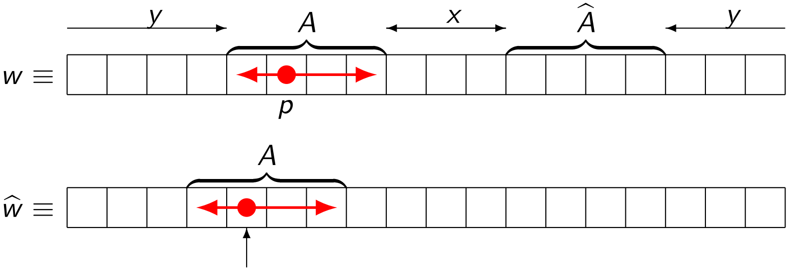

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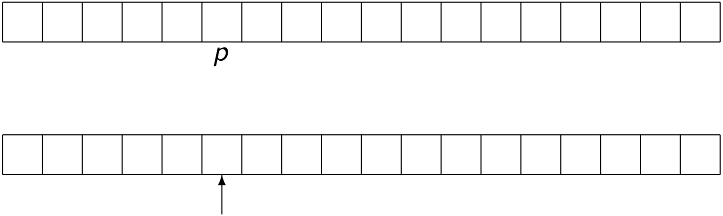

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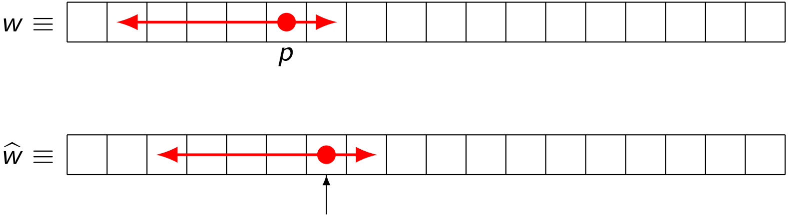

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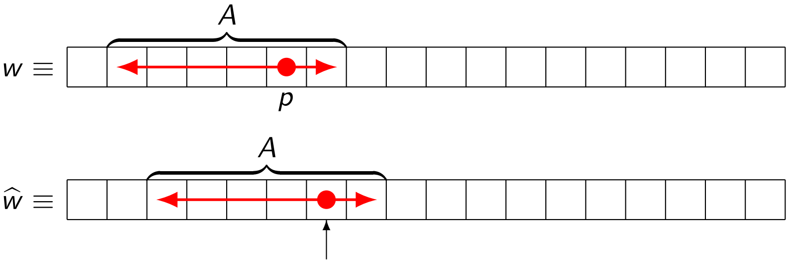

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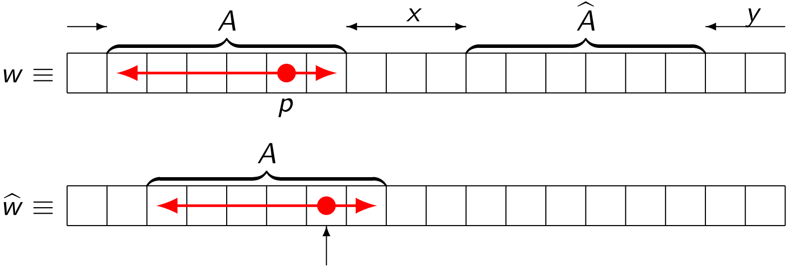

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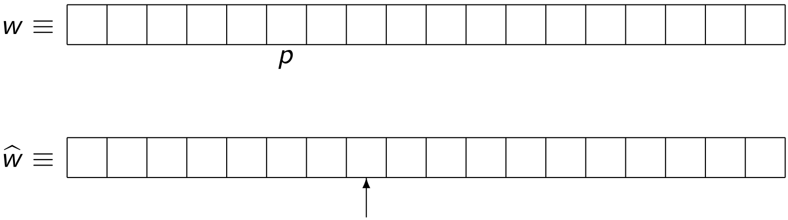

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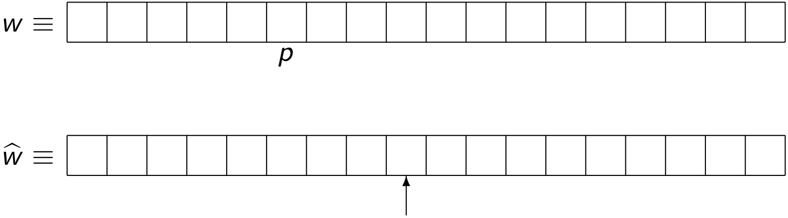

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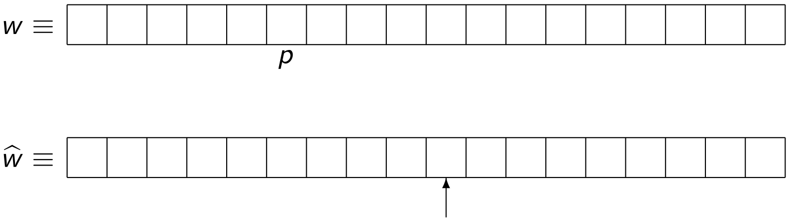

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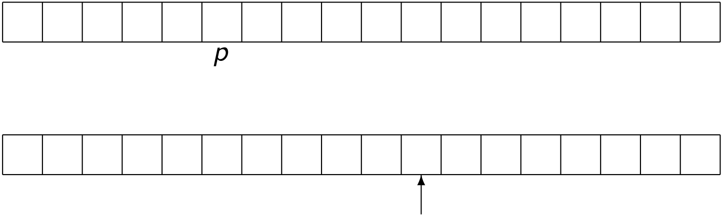

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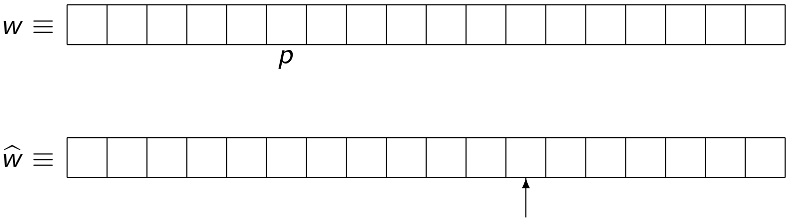

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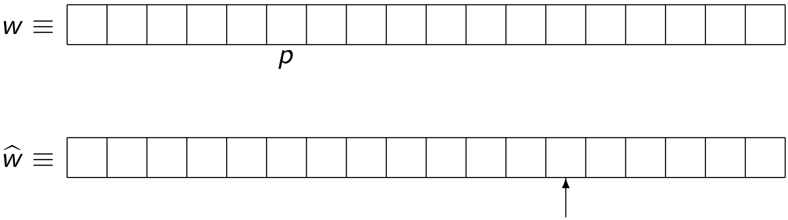

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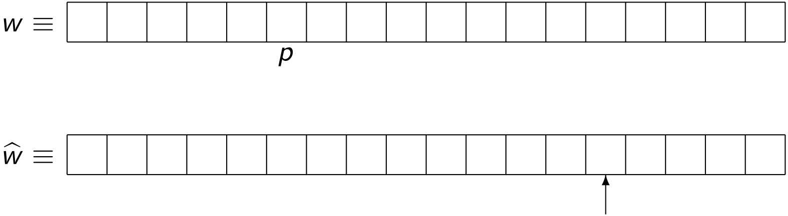

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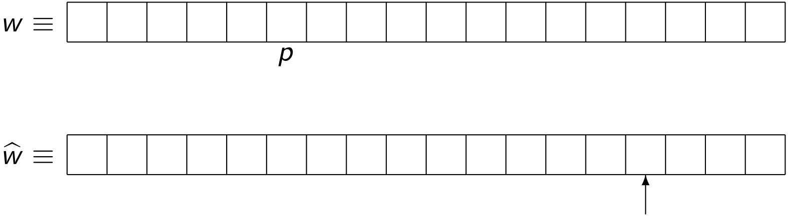

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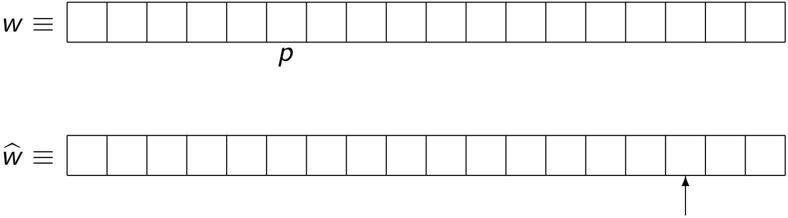

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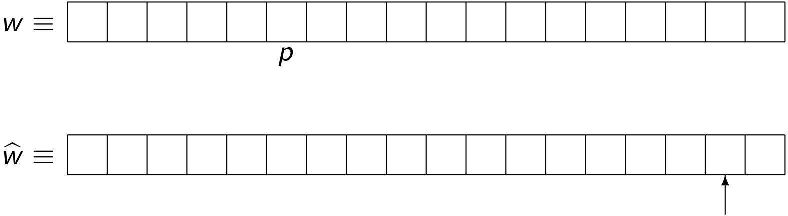

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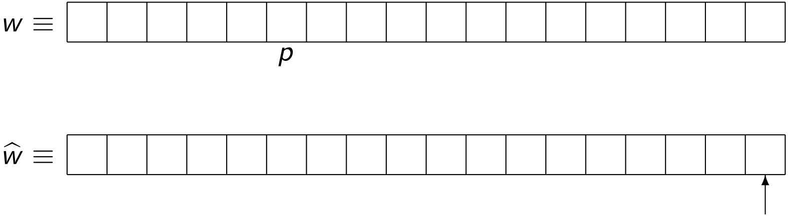

The algorithms

Detecting pseudo-squares

Theorem (Brlek and Provençal, 2006)

Let w be the boundary of p. Determining if w codes a pseudo-square is decidable in linear time.

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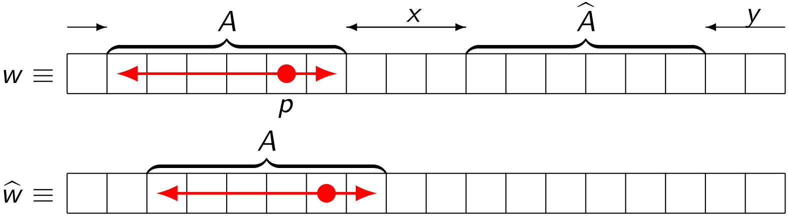

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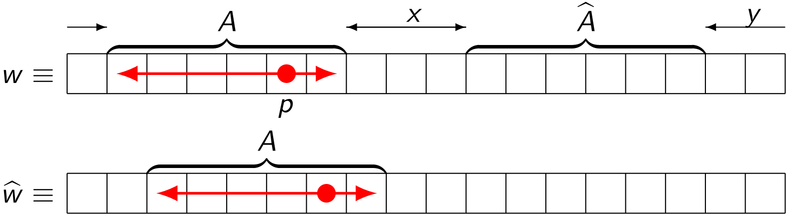

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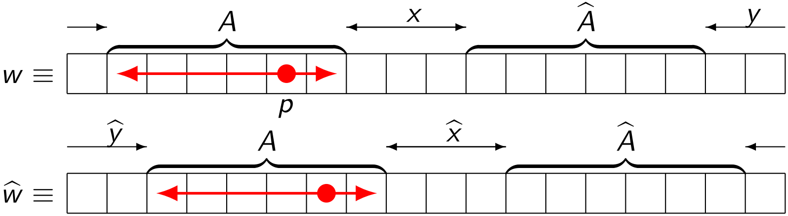

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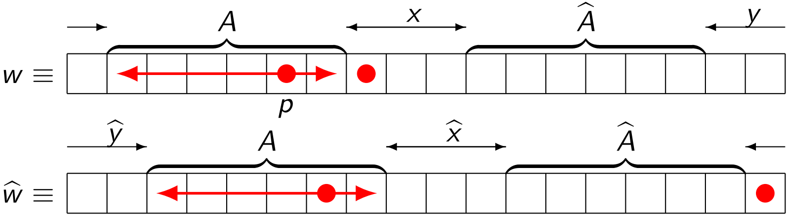

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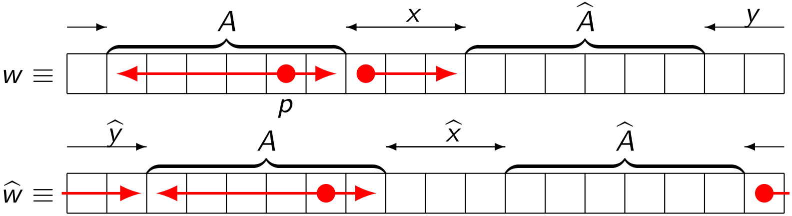

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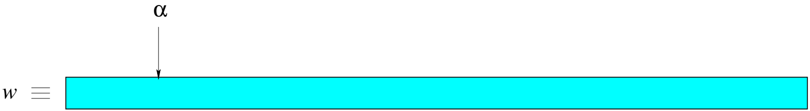

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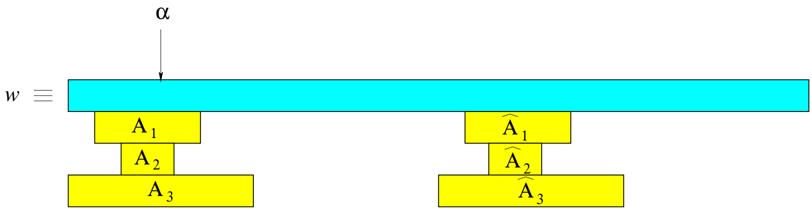

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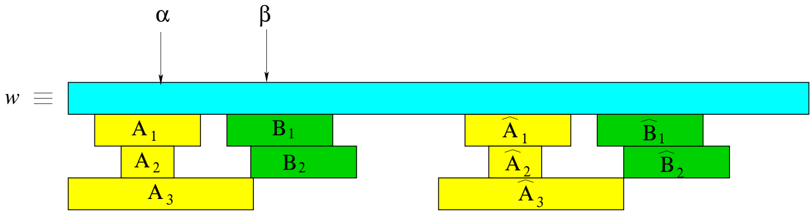

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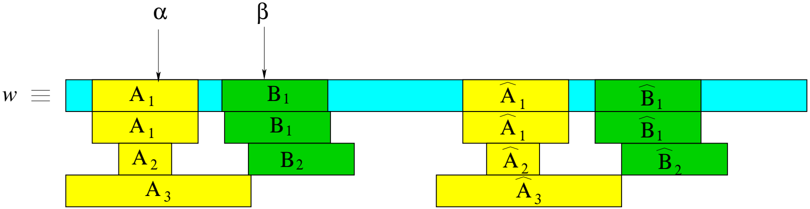

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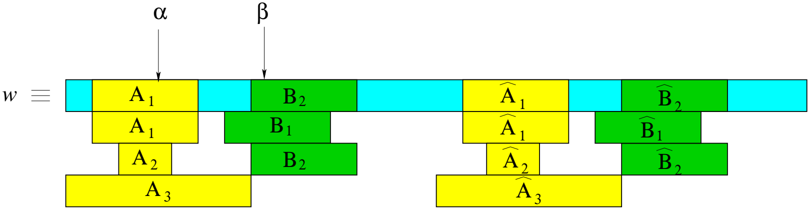

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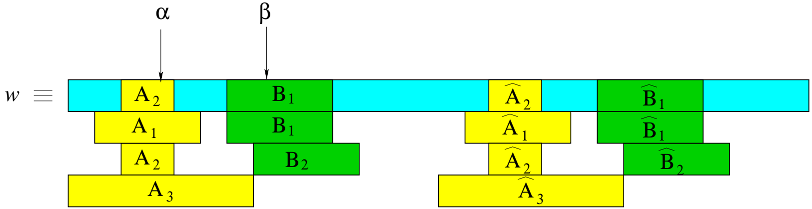

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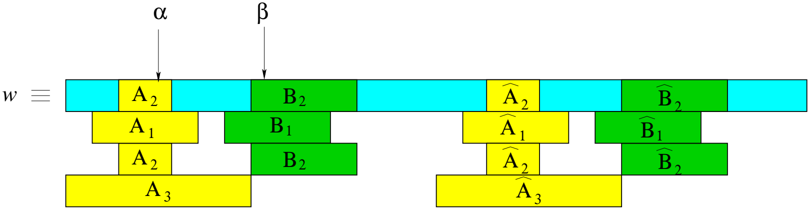

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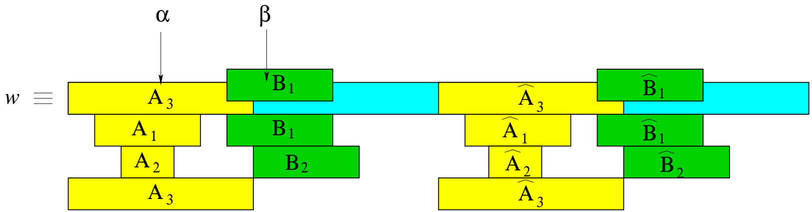

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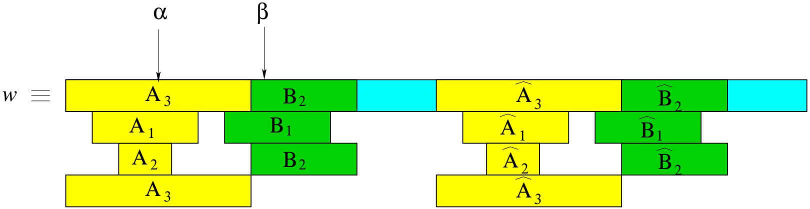

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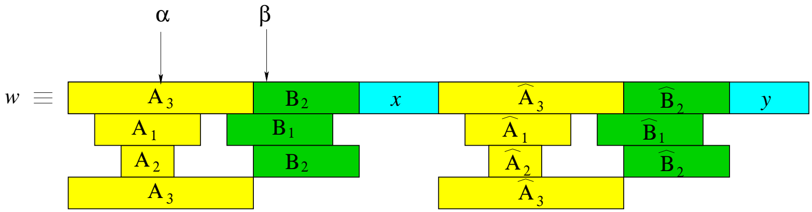

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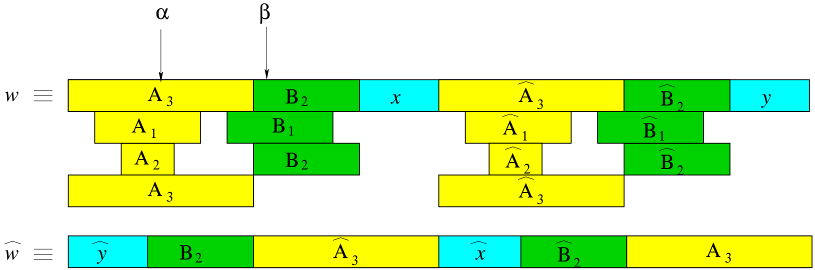

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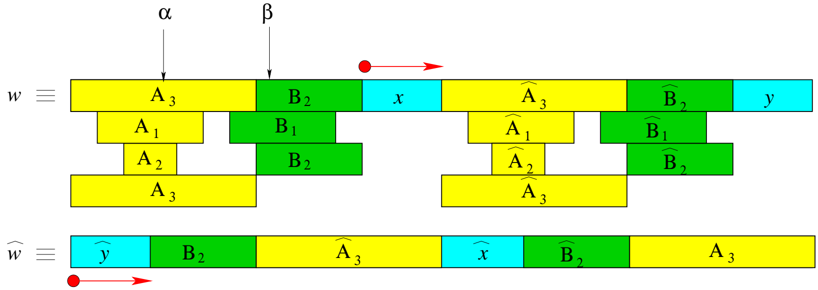

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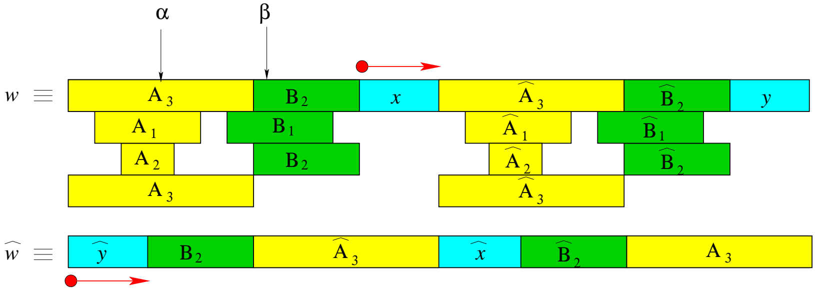
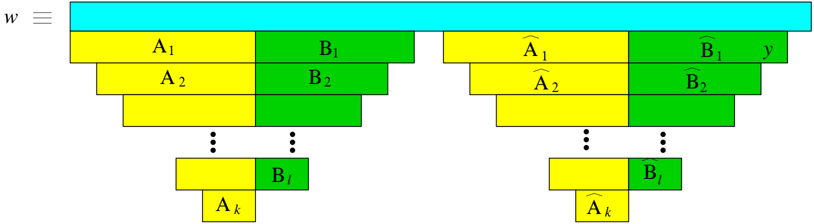


Theorem (Brlek and Provençal, 2006)

Given a contour word w of length n such that its longest factor of the form xx is shorter than $\sqrt{n}$, then this algorithm tests if it is a pseudo-hexagon in $\mathcal{O}(n)$.

Worst case



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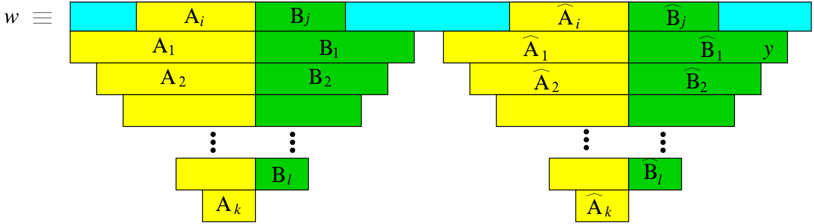
$w \equiv$
A_i
B_j
$\widehat{A}_i$
$\widehat{B}_j$
A_1
B_1
$\widehat{A}_1$
$\widehat{B}_1$
y
A_2
B_2
$\widehat{A}_2$
$\widehat{B}_2$
B_l
$\widehat{B}_l$
A_k
$\widehat{A}_k$

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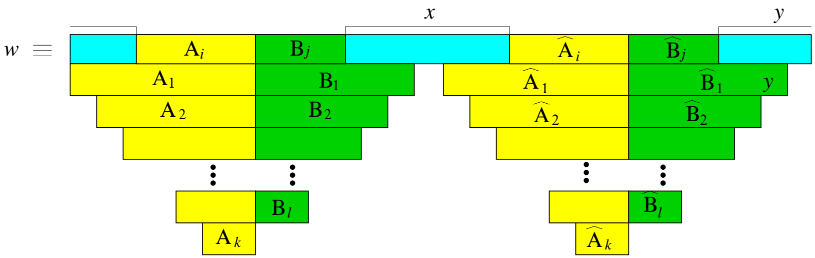
x
y
w ≡
A_i
B_j
$\widehat{A}_i$
$\widehat{B}_j$
A_1
B_1
$\widehat{A}_1$
$\widehat{B}_1$
y
A_2
B_2
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$\widehat{B}_2$
B_l
$\widehat{B}_l$
A_k
$\widehat{A}_k$

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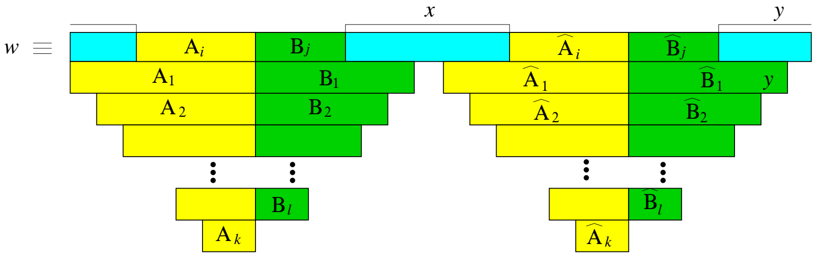
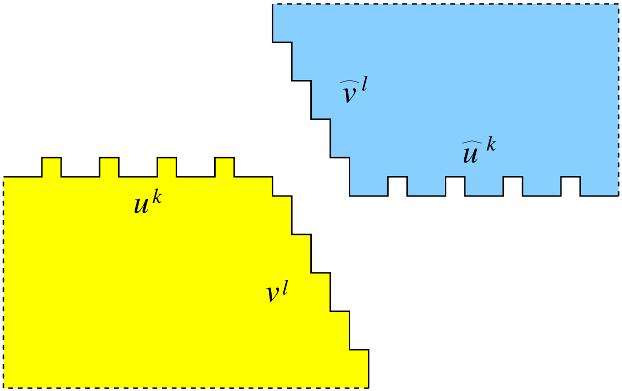
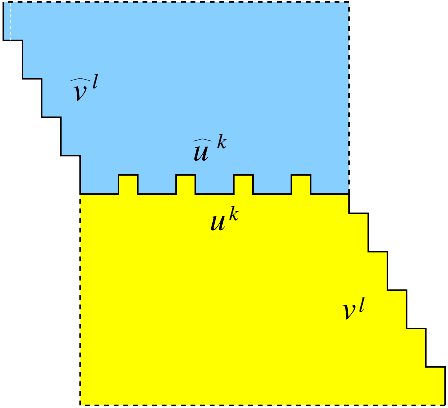


Remark

Since k and l can be in $\mathcal{O}(n)$ the number of tests may raise to $\mathcal{O}(n^2)$.


$\widehat{v}^l$
$\widehat{u}^k$
u^k
v^l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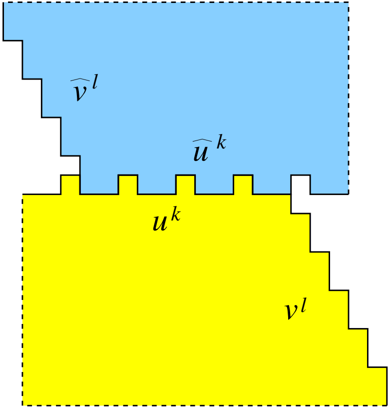
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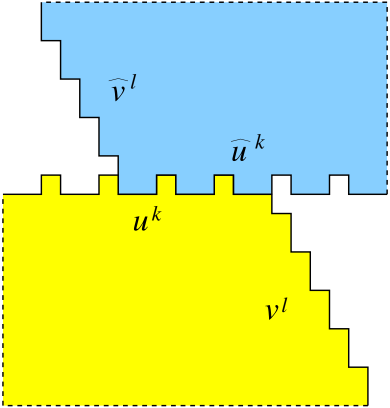
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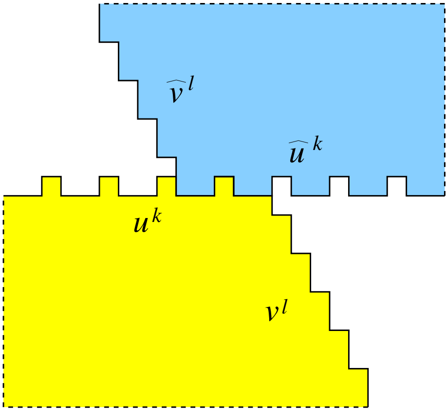
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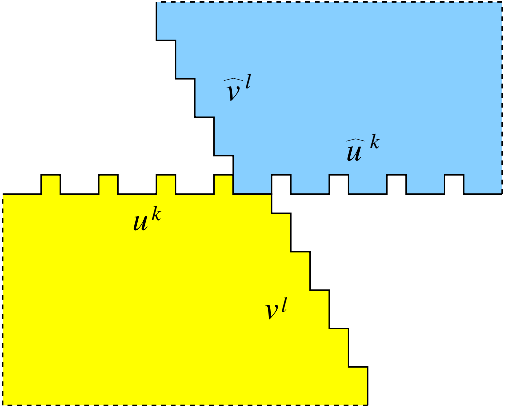
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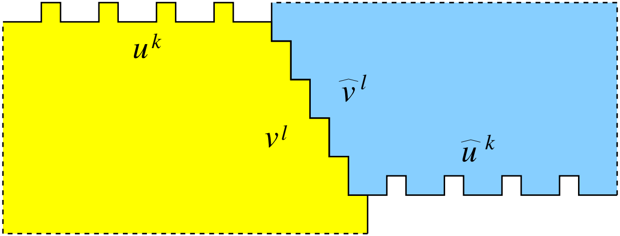
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v^l
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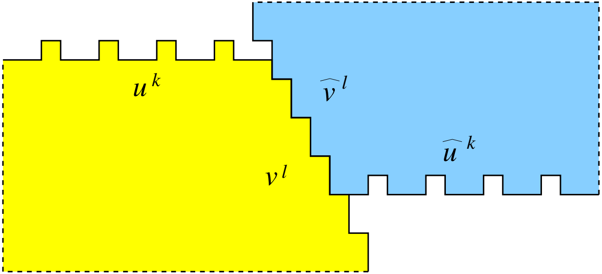
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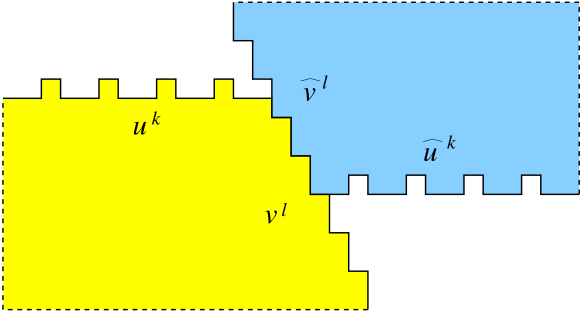
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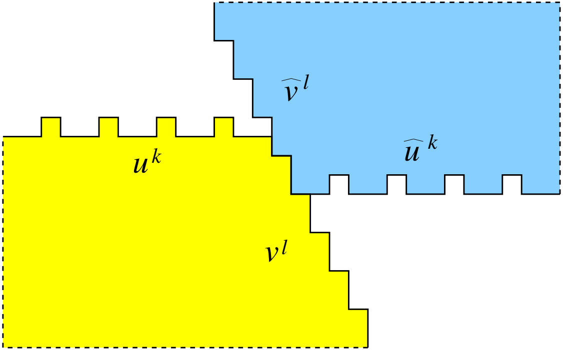
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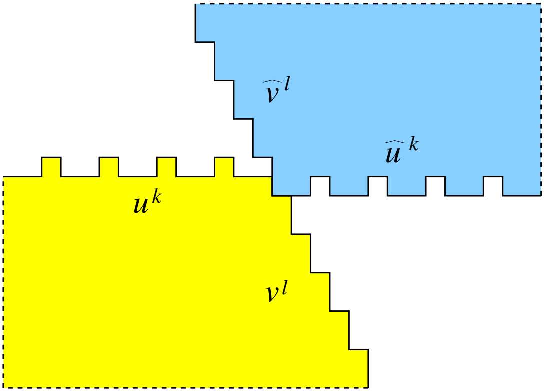
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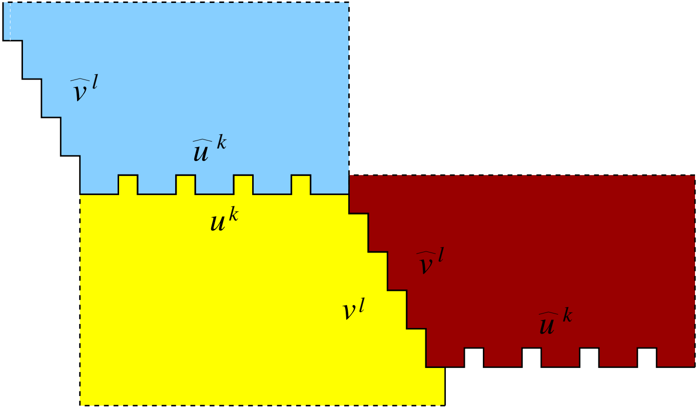
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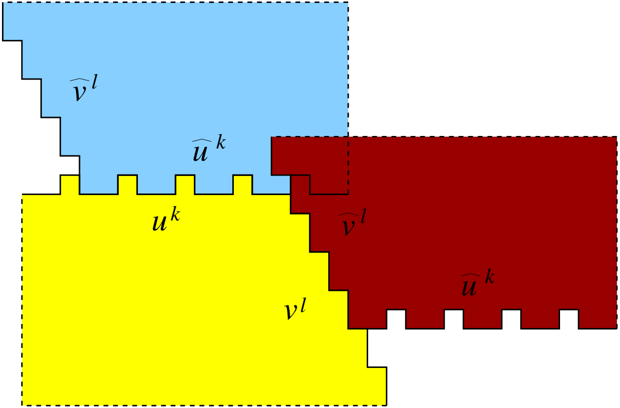
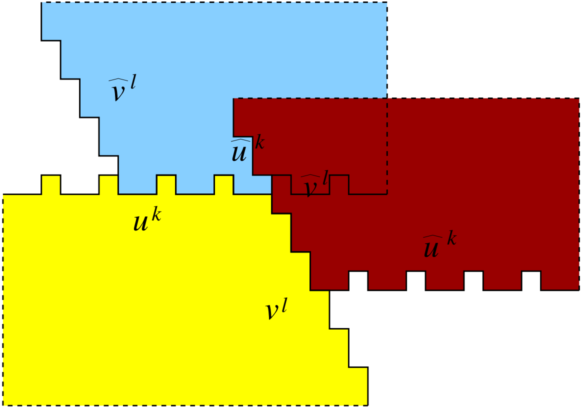
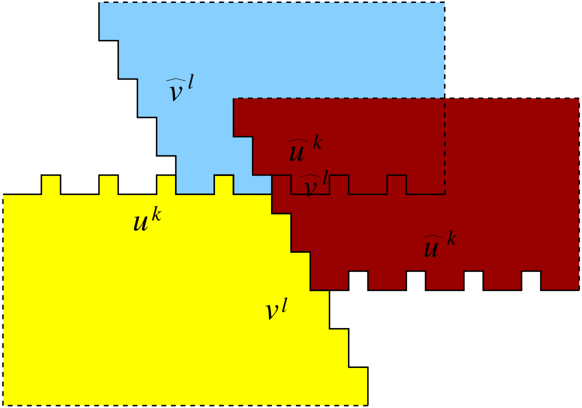
u^k
$\widehat{v}^l$
v^l
$\widehat{u}^k$


$\widehat{v}^l$
$\widehat{u}^k$
u^k
$\widehat{v}^l$
$\widehat{u}^k$
v^l


$\widehat{v}^l$
$\widehat{u}^k$
$\widehat{v}^l$
u^k
$\widehat{u}^k$
v^l


$\widehat{v}^l$
$\widehat{u}^k$
$\widehat{v}^l$
u^k
$\widehat{u}^k$
v^l

Periodicity in words

Definition

A word $w = w_1 w_2 \cdots w_n$ *has a period* $p \geq 1$ *if* $w_i = w_{i+p}$ *for all* $i \in \{1, 2, \ldots, n-p\}$.

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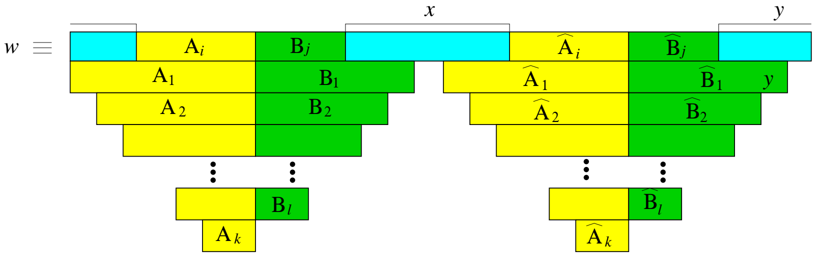
Theorem (Fine and Wilf, 1965)

If a word w *admit two periods* p *and* q *such that* $|w| \geq p + q - \gcd(p, q)$ *then* $\gcd(p, q)$ *is also a period of* w.

Worst case


x
y
w ≡
A_i
B_j
$\widehat{A}_i$
$\widehat{B}_j$
A_1
B_1
$\widehat{A}_1$
$\widehat{B}_1$
y
A_2
B_2
$\widehat{A}_2$
$\widehat{B}_2$
B_l
$\widehat{B}_l$
A_k
$\widehat{A}_k$

Theorem (Provençal, 2008)

Testing if a polyomino tiles the plane can be done in $\mathcal{O}(n(\log n)^3)$ time.

MERCI!