An Optimal Algorithm for Detecting Pseudo-Squares

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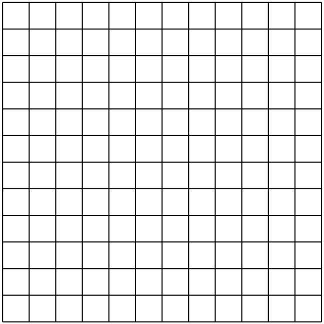
Université du Québec à Montréal,

October 25, 2006

Introduction to polyominos

- Discrete plane : $\mathbb{Z}^2$



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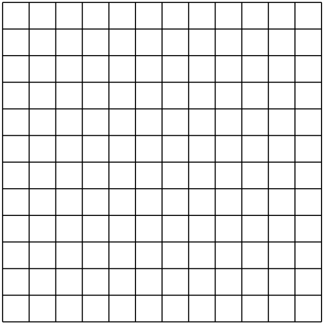

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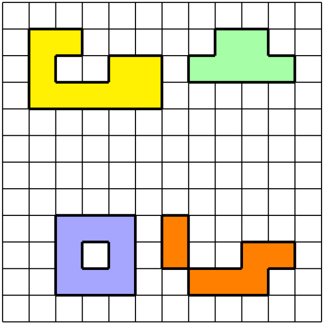

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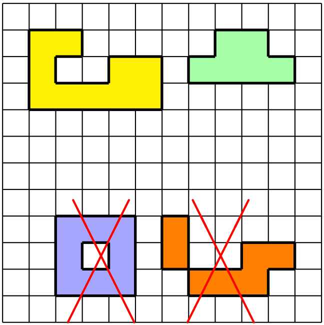

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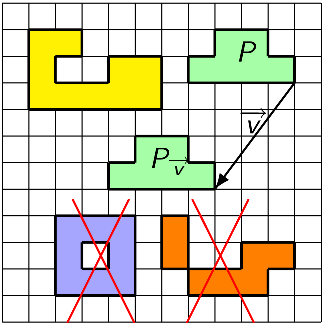
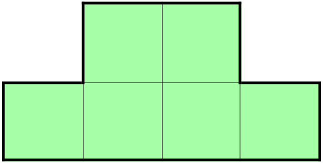
- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A *polyomino* is a finite, 4-connected subset of the plane, without holes.
- **Notation** : Let P be a polyomino and $\overrightarrow{v}$ a vector o $\mathbb{Z}^2$, $P_{\overrightarrow{v}}$ will denote the image of P by de translation $\overrightarrow{v}$.



Freeman chain code

$$\Sigma = \{a, \overline{a}, b, \overline{b}\}$$



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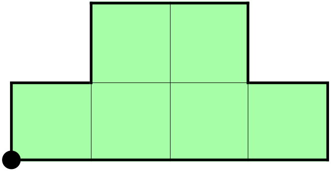

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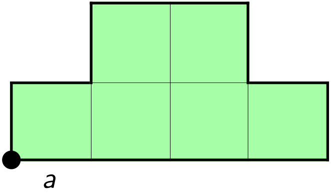

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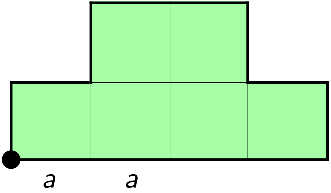

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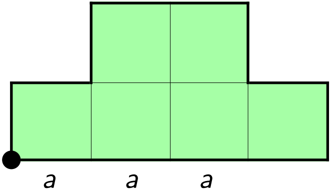

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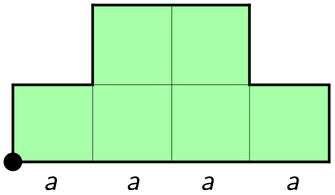

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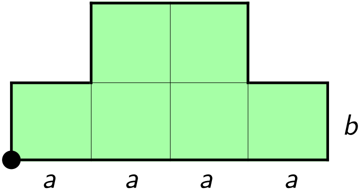

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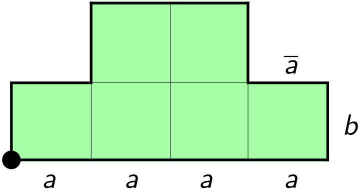

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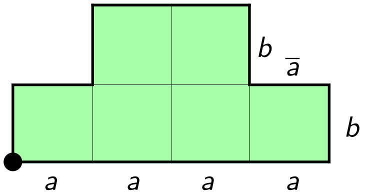

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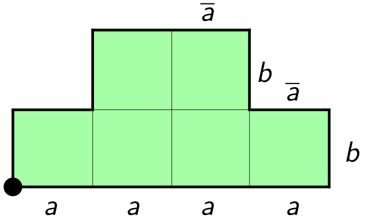

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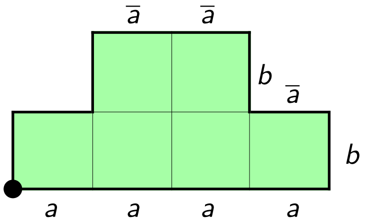

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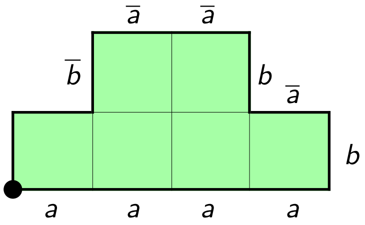

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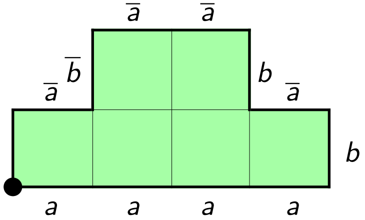

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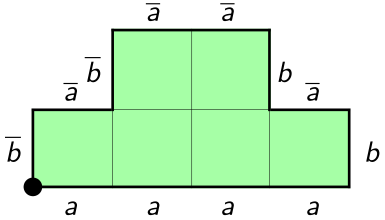

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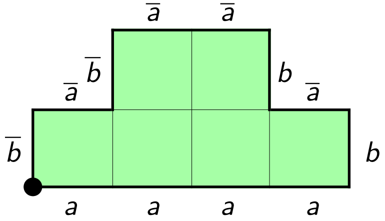

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$w \equiv w'$ notes that w and w' are conjugate.

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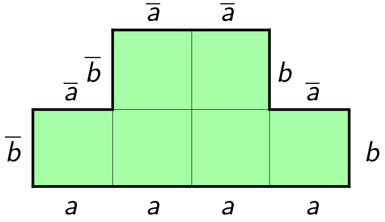

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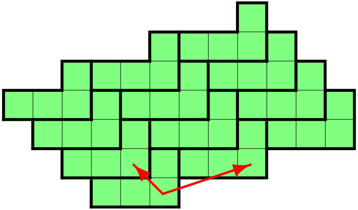

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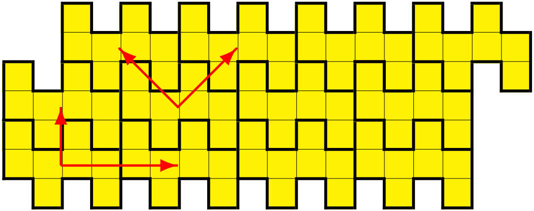

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Upper bound : ? ? ?

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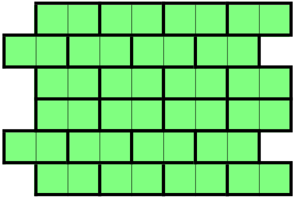
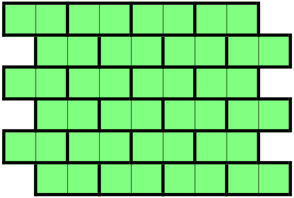
Irregular $\Longrightarrow$ Regular





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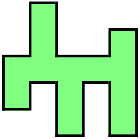

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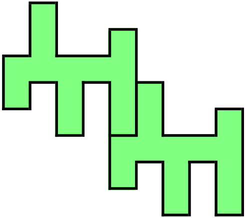

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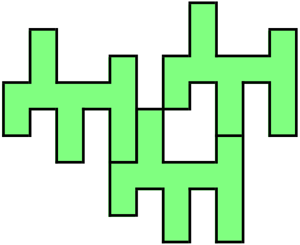

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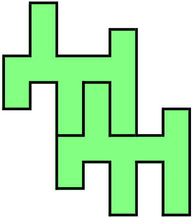

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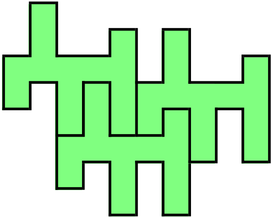

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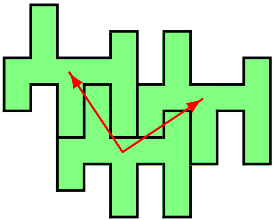

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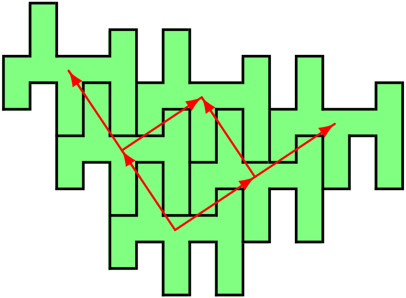

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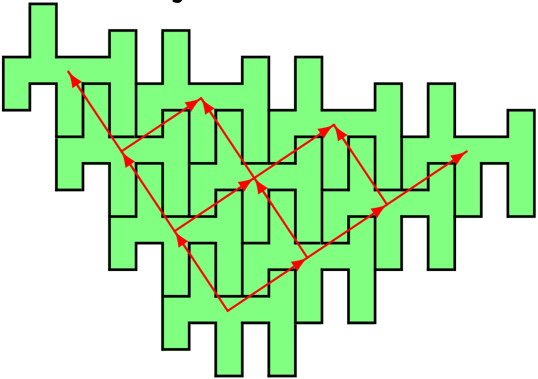

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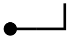

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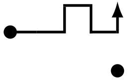

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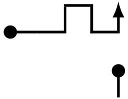

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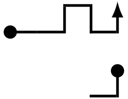

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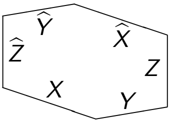
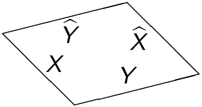


Pseudo-squares



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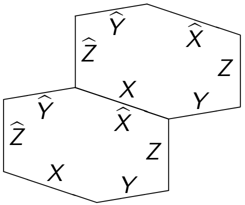

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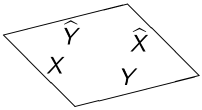

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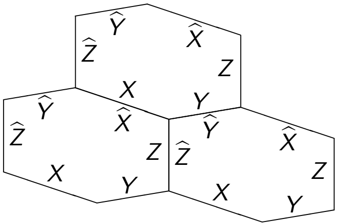

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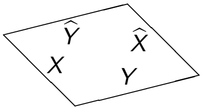

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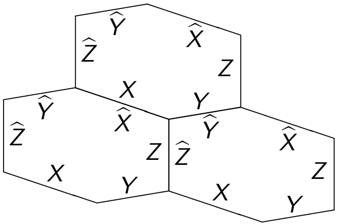

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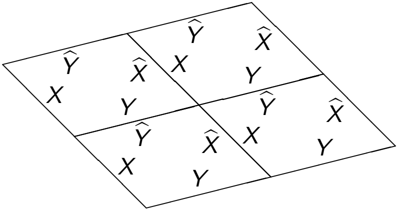

Gambini and Vuillon

2003 - Gambini and Vuillon

$\mathcal{O}(n^2)$ algorithm using Beauquier-Nivat's characterization.

Admissible factors

Definition

Let A be a factor of the word w coding a polyomino p. A is admissible if

- $w \equiv Ax\widehat{A}y$, *for* x, y *such that* $|x| = |y|$.
- *A is saturated, that is,* $\text{first}(x) \neq \overline{\text{last}(x)}$ *and* $\text{first}(y) \neq \overline{\text{last}(y)}$.

Admissible factors

Proposition

Let $\mathcal{A}$ be the set of all admissible factors overlapping a position α in w and $\widehat{\mathcal{A}}$ be the set of their respective homologue factors. Then, there is at least one position in w that is not covered by any element of $\mathcal{A} \cup \widehat{\mathcal{A}}$.

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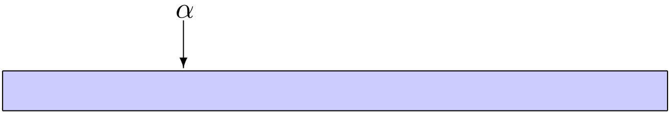

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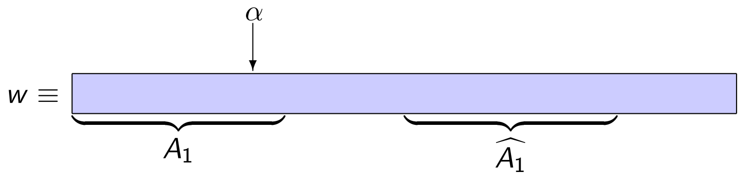

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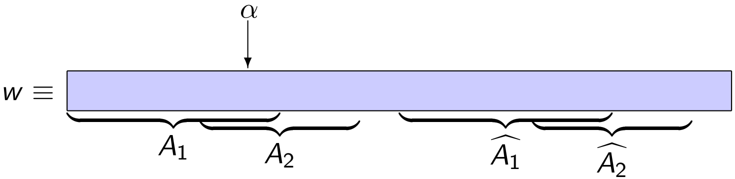

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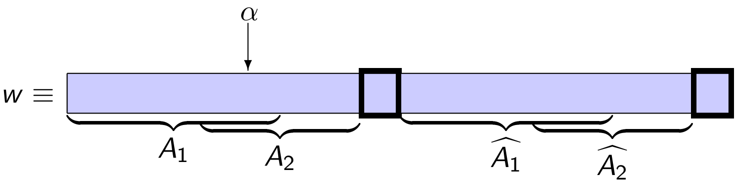

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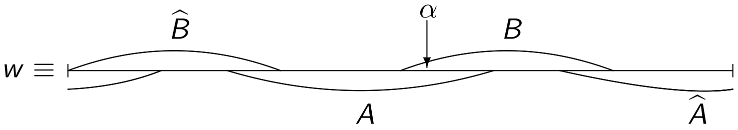

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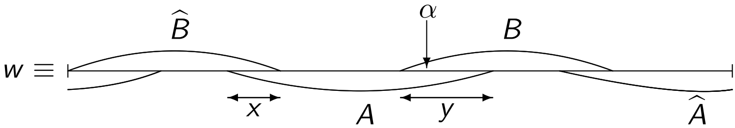

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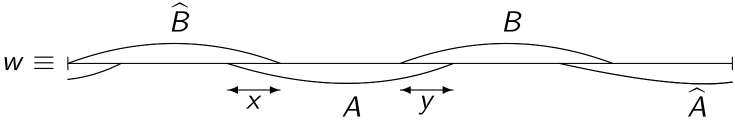

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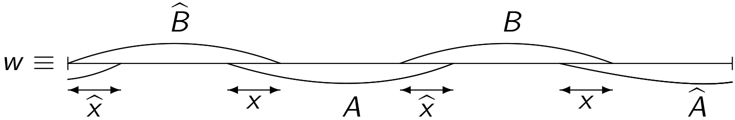

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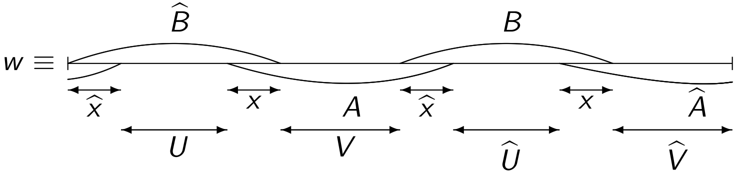

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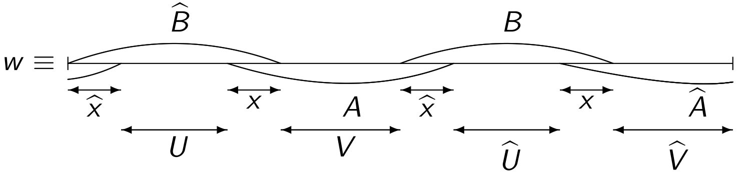

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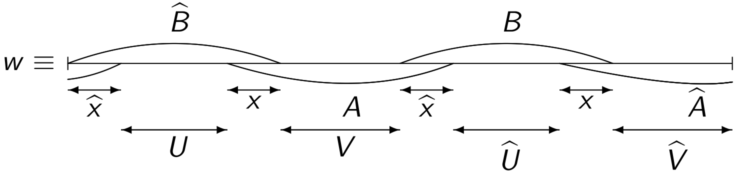

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In a non-intersecting closed path on a square lattice,

$$\#(\mathit{left\ turns}) - \#(\mathit{right\ turns}) = 4.$$

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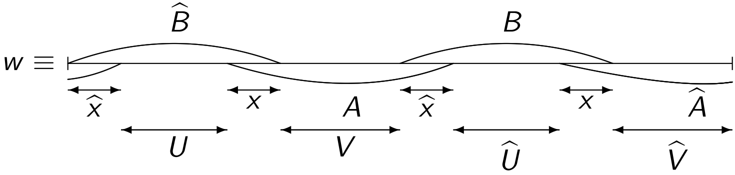

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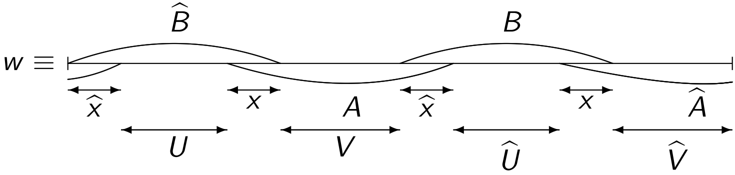

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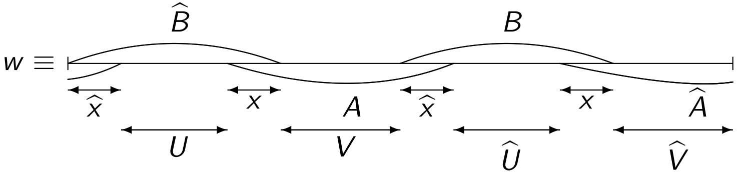

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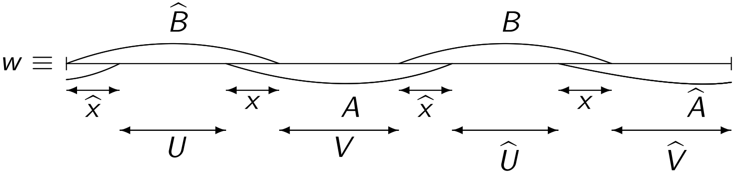

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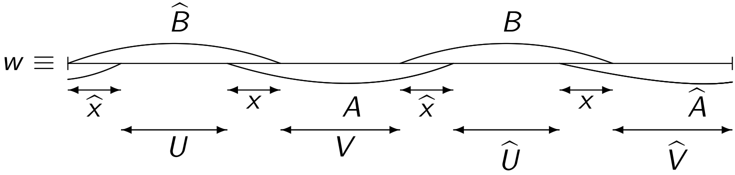

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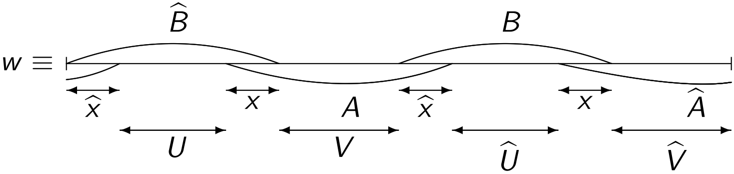

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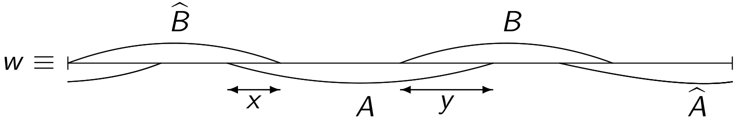

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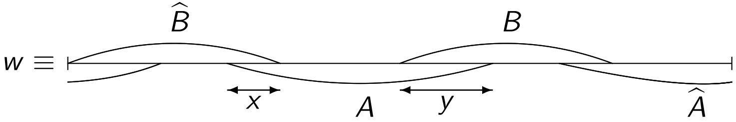

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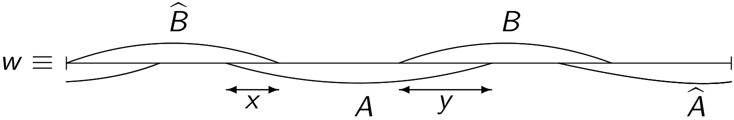

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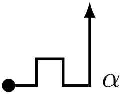

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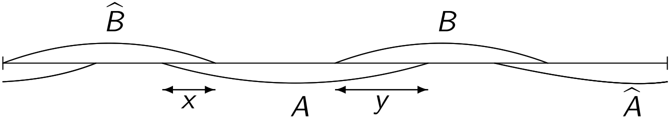
$w \equiv$ 

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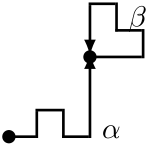

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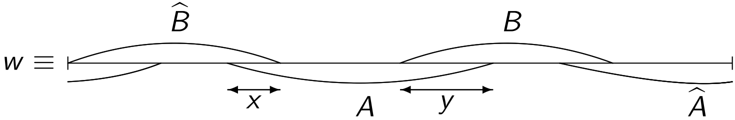

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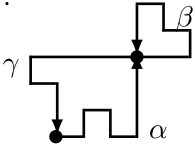

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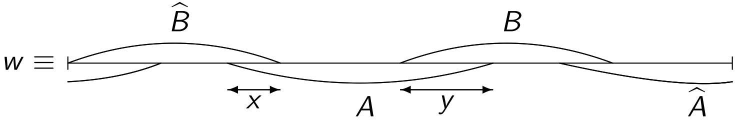

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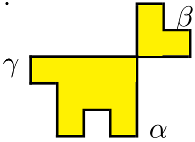

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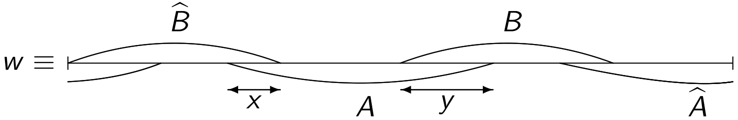

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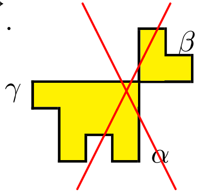

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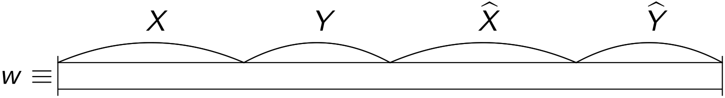

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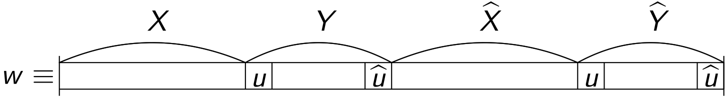

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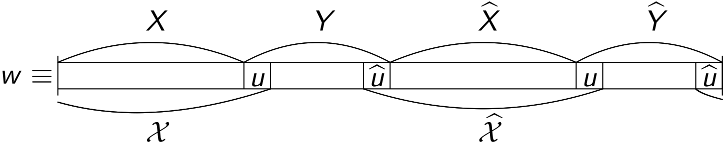

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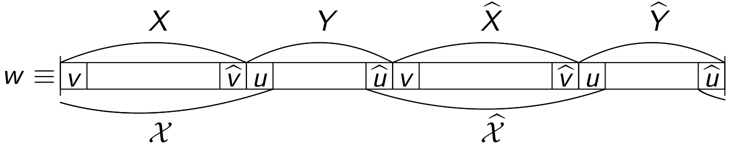

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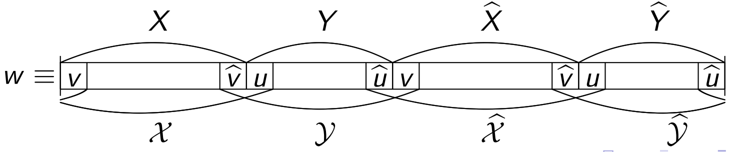

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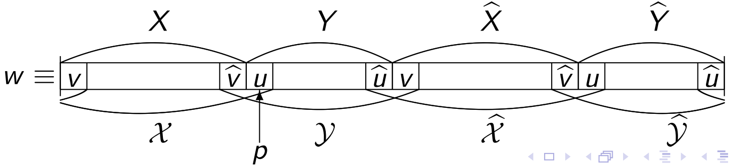

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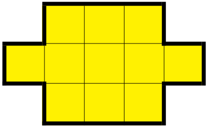

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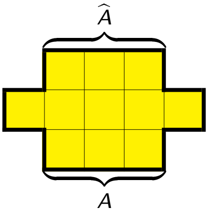

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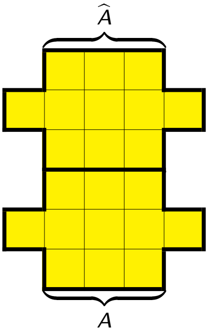

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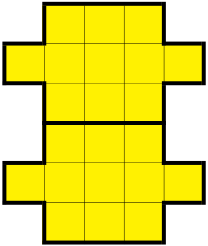

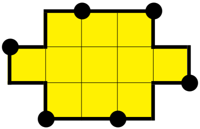

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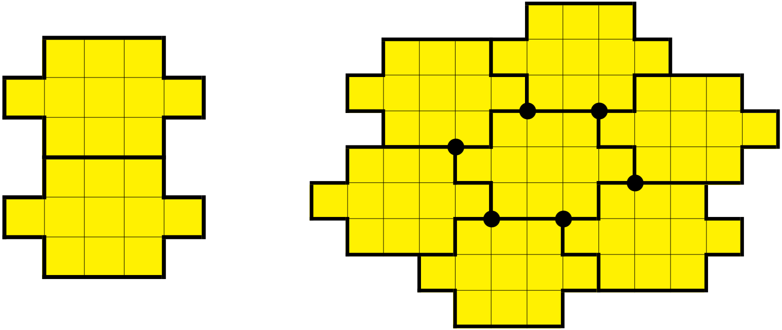

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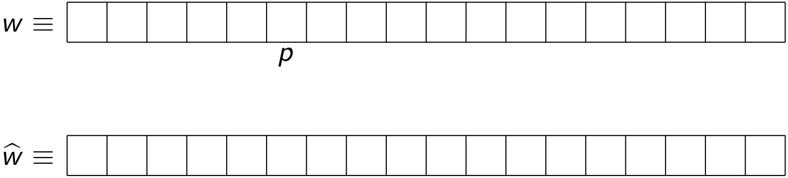

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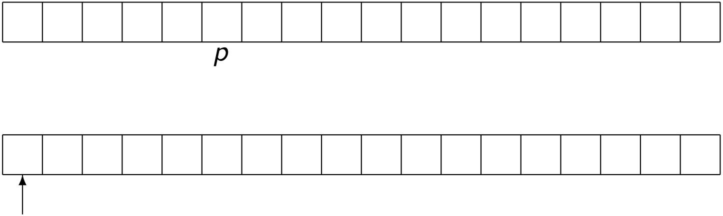

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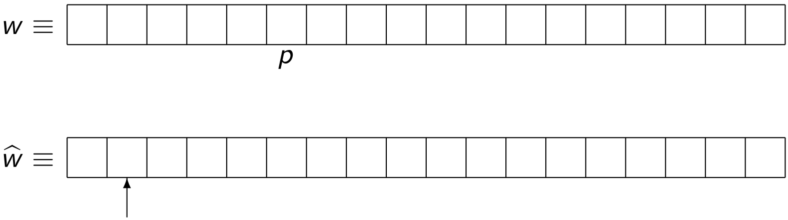

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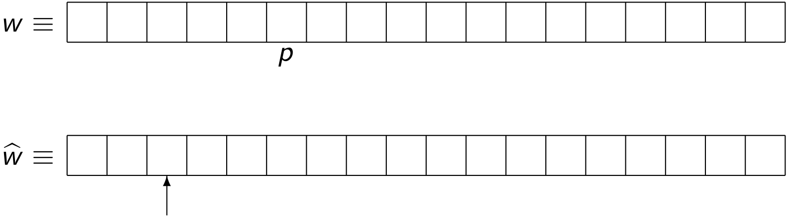

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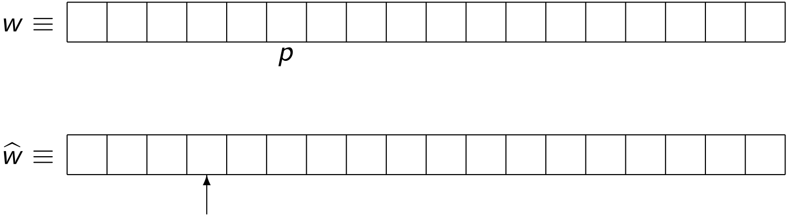

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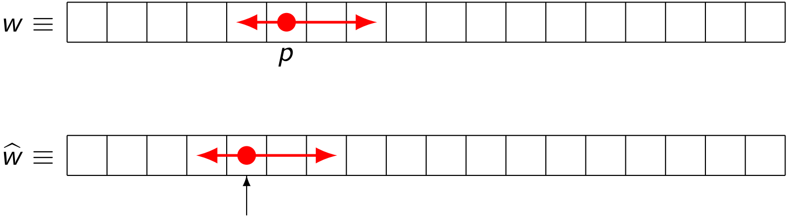

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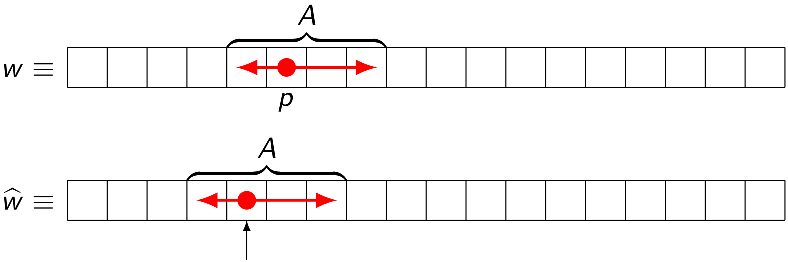

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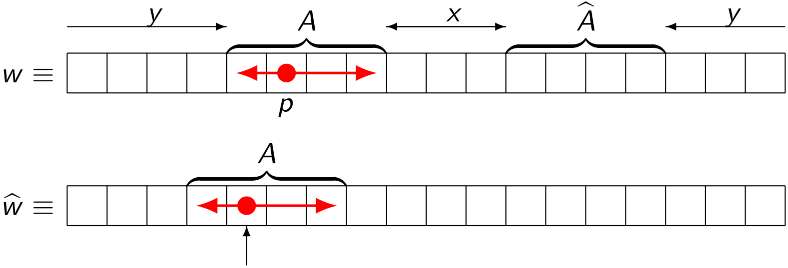

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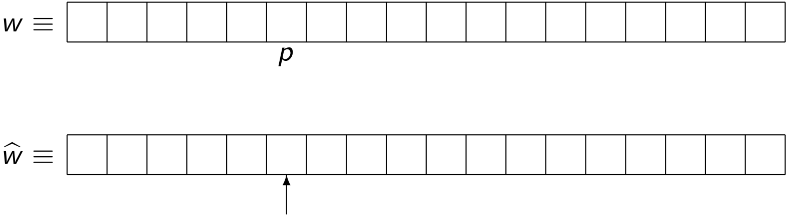

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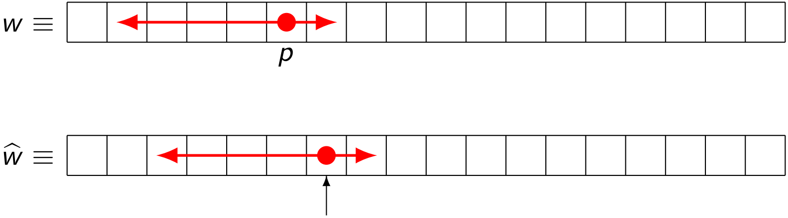

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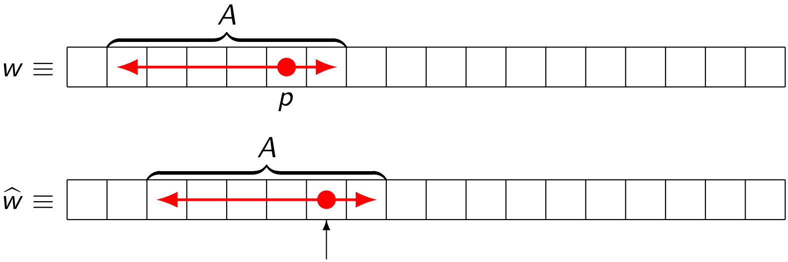

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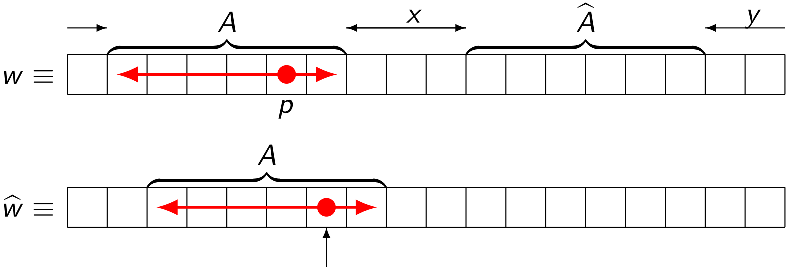

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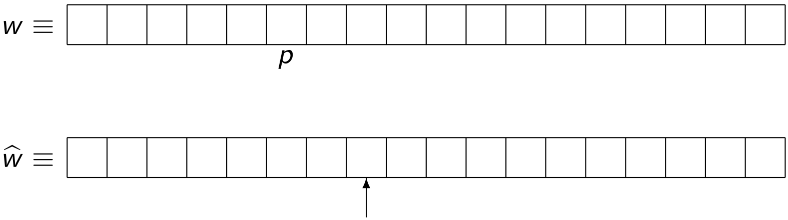

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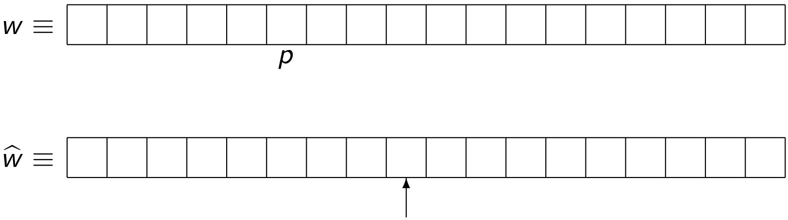

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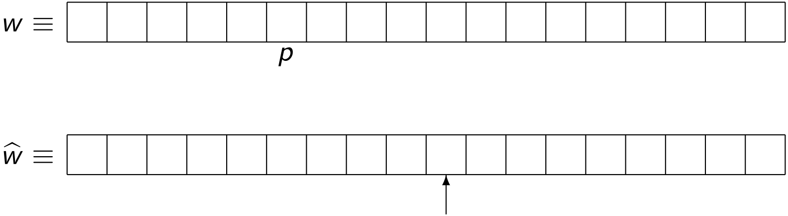

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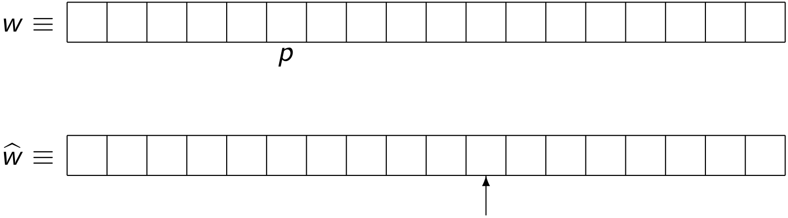

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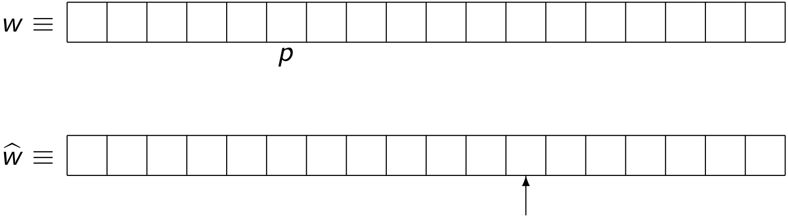

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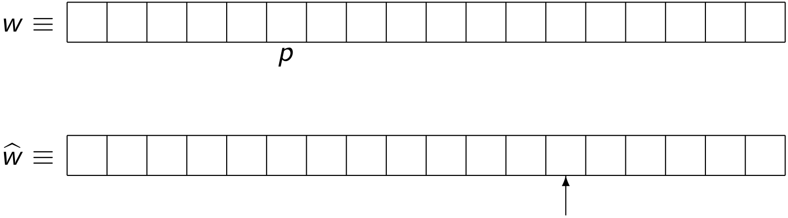

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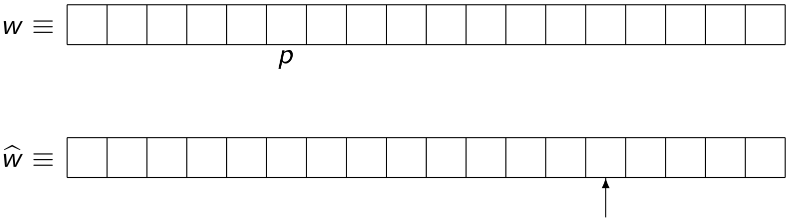

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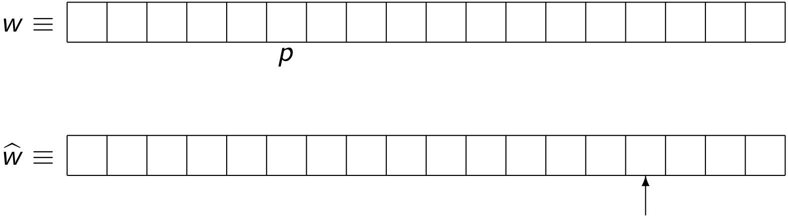

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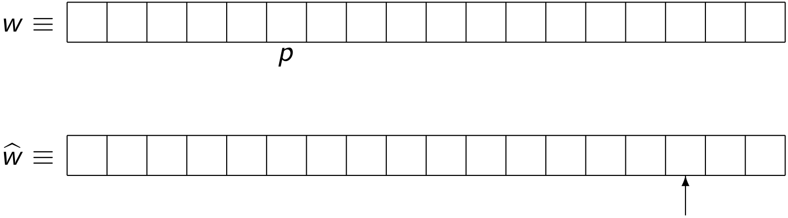

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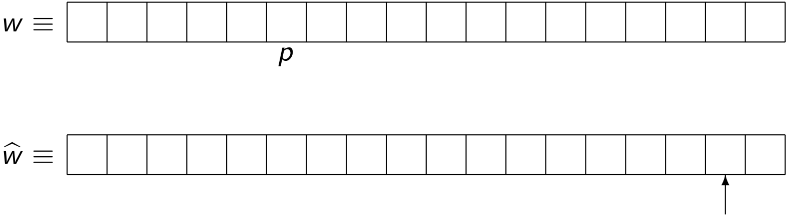

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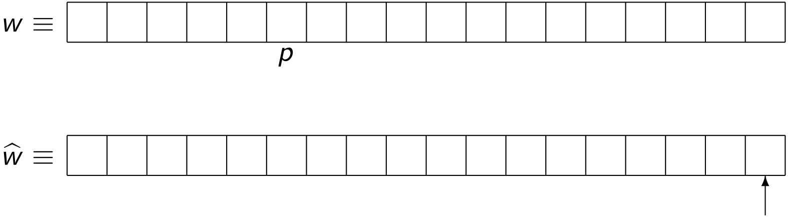

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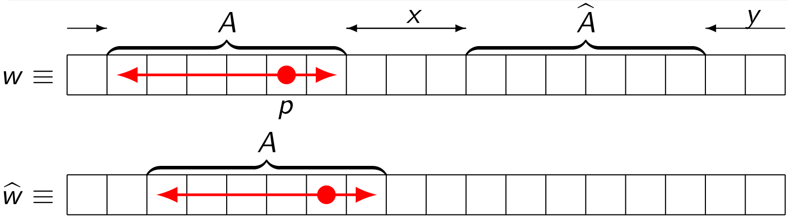

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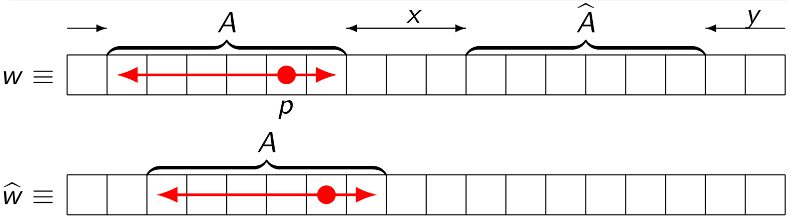

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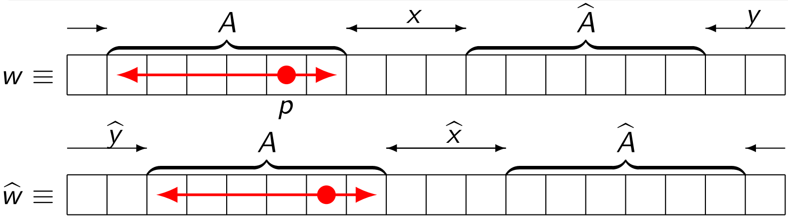

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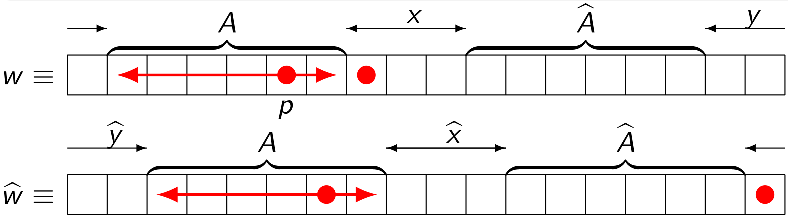

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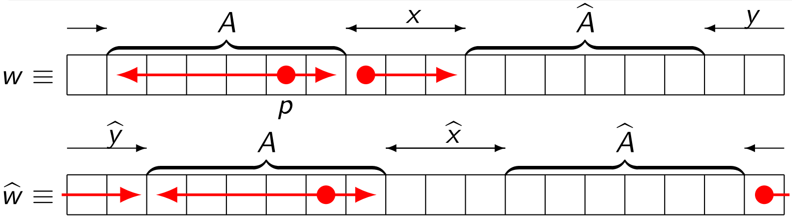

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