On the problem of tiling the plane with a polyomino

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12 mars, 2006

Outline

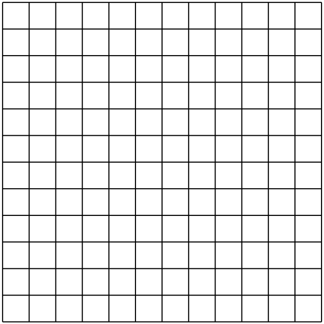
1 The tiling problem

2 Beauquier-Nivat characterization

3 A fast algorithm to detect exact polyominoes

Introduction to polyominoes

- Discrete plane : $\mathbb{Z}^2$



Introduction to polyominoes

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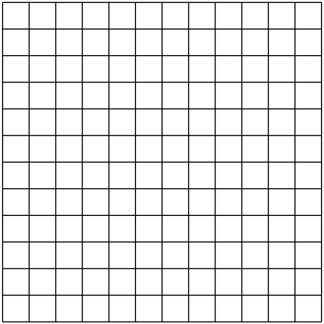
- **Definition** : A *polyomino* is a finite, 4-connected subset of the plane, without holes.



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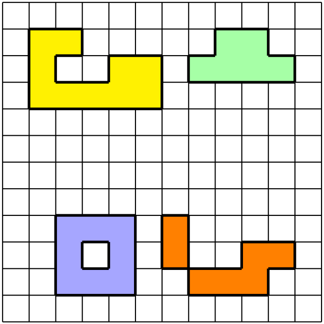

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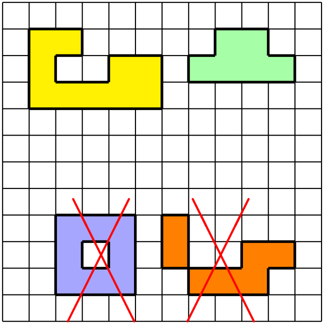

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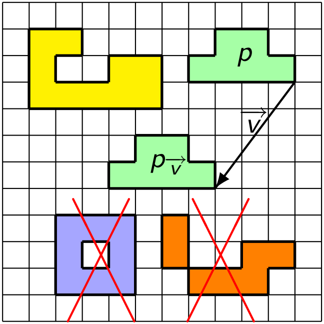
- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A *polyomino* is a finite, 4-connected subset of the plane, without holes.
- **Notation** : Let p be a polyomino and $\overrightarrow{v}$ a vector of $\mathbb{Z}^2$, $p_{\overrightarrow{v}}$ will denote the image of p by de translation $\overrightarrow{v}$.



General statement of the tiling problem

Definition (Tiling)

A tiling $\mathcal{T}$ of a subset $D \subset \mathbb{Z}^2$ by a set of polyominoes $\mathcal{P}$ is a set of couples $(p, \overrightarrow{u}) \in \mathcal{P} \times \mathbb{Z}^2$ such that :

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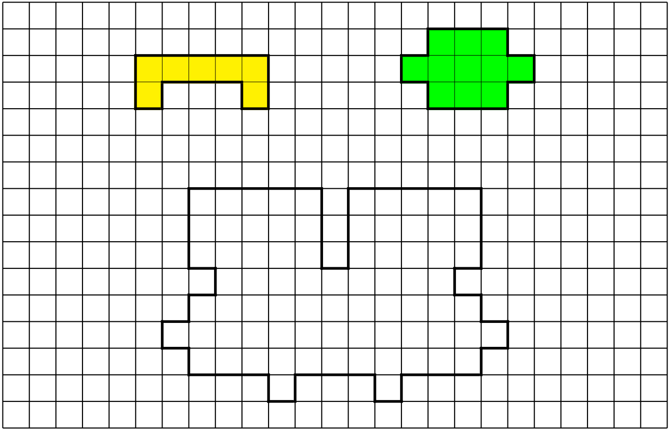
Definition (The Tiling Problem)

Given a set of polyominoes $\mathcal{P}$ and a subset $D \subset \mathbb{Z}^2$.
Does D admits a tiling by $\mathcal{P}$.

Example



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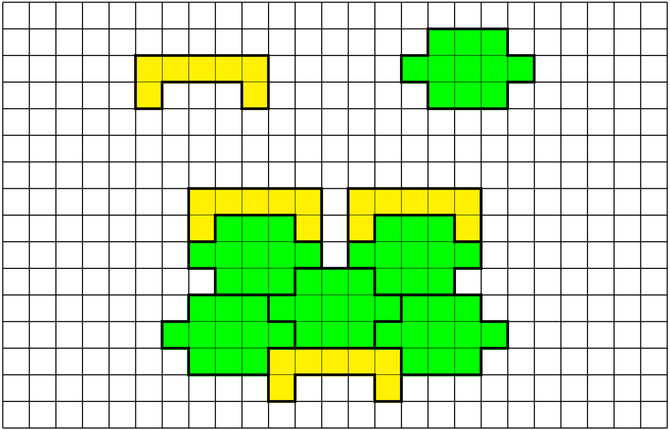

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Theorem (Garey, Johnson and Papadimitriou)

The tiling problem with D finite and $\mathcal{P} = \{$  ,  $\}$ *is NP-Complete.*

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We concider the case where $D = \mathbb{Z}^2$ and $\mathcal{P}$ is finite.

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A tiling $\mathcal{T}$ is half-periodic if there exists a vectors $\overrightarrow{u}$ such that $\mathcal{T}$ is not changed by the corresponding translation.

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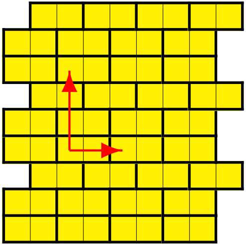

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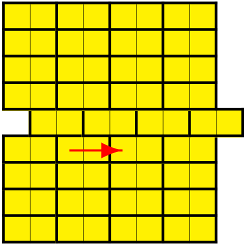

Half-periodic tiling

Half-Periodic implies periodic

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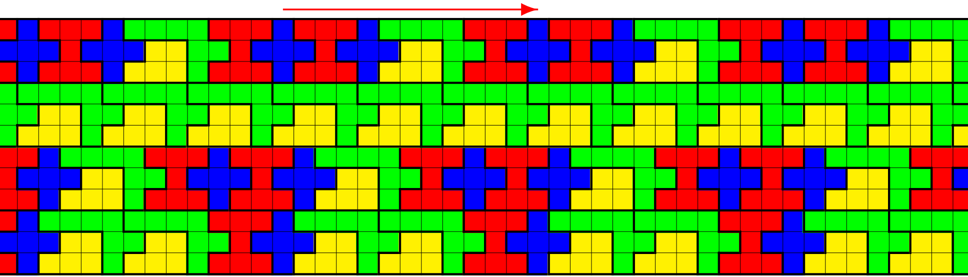

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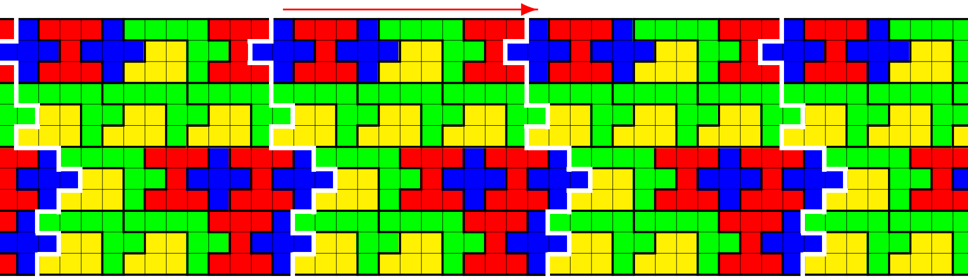

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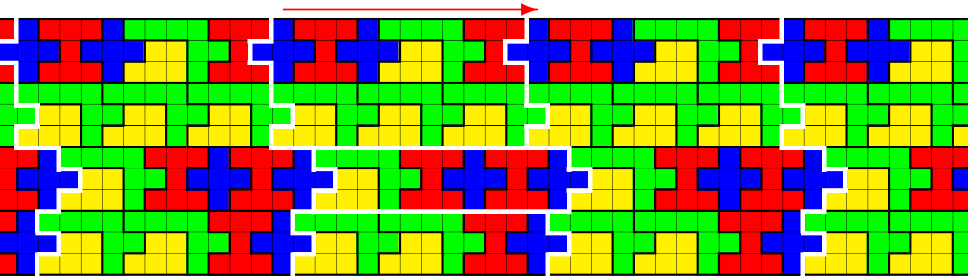

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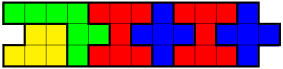

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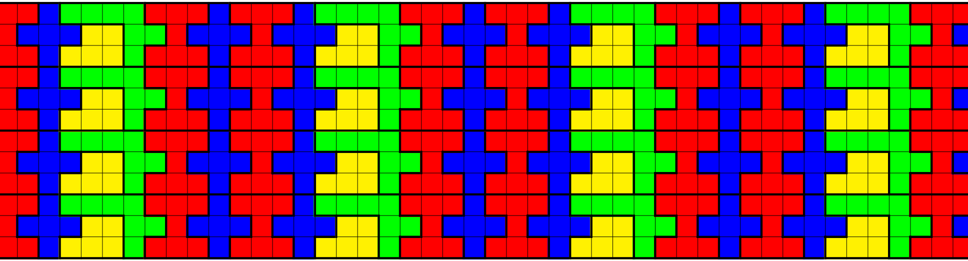

Nonperiodic tilings

Theorem (Berger, 1966)

The tiling problem with $\mathcal{P}$ finite and $D = \mathbb{Z}^2$ is undecidable.

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Corollary

There are some finite sets $\mathcal{P}$ such that tilings of the plane by $\mathcal{P}$ do exist and are all nonperiodic.

Tilings with one polyomino

Definition

A polyomino p is exact if the set $\mathcal{P} = \{p\}$ *tiles the plane.*

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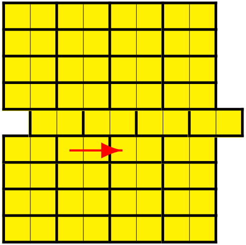
Definition

A tiling of the plane $\mathcal{T}$ *by an exact polyomino p is regular if there exist two vectors* $\overrightarrow{u}$ *and* $\overrightarrow{v}$ *such that*

$$\mathcal{T} = \{(p, i\overrightarrow{u} + j\overrightarrow{v}) | i, j \in \mathbb{Z}^2\}$$

Examples



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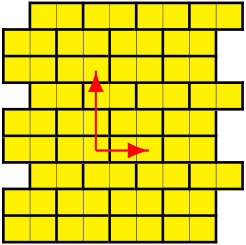
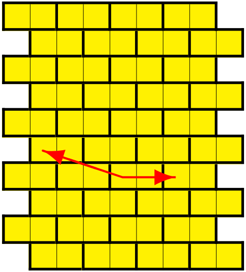


Periodic tiling



Regular tiling

Tilings with one polyomino

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If a polyomino p tiles the plane, then there exists a regular tiling of the plane by p.

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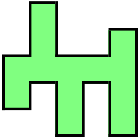
Corollary

The tiling problem with $\mathcal{P} = \{p\}$ and $D = \mathbb{Z}^2$ is decidable in polynomial time.

Example



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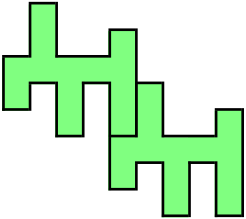

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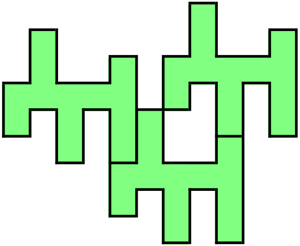

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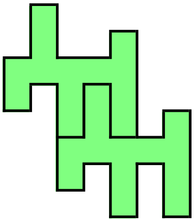

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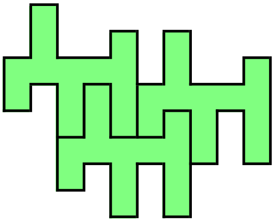

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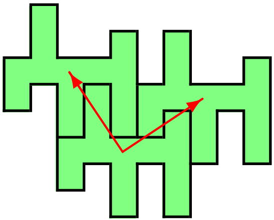

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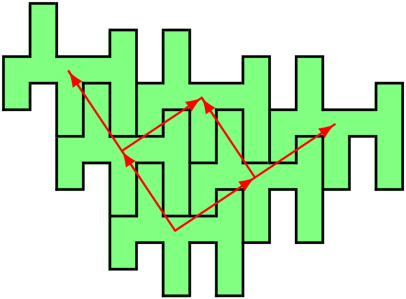

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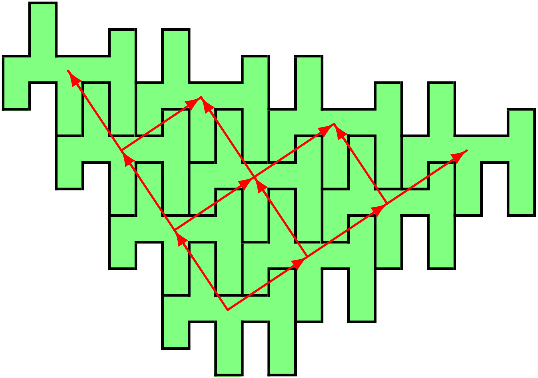

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3. A fast algorithm to detect exact polyominoes

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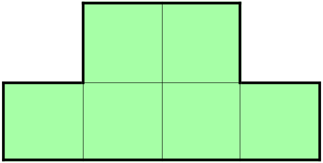
$$\Sigma = \{a, \bar{a}, b, \bar{b}\}$$



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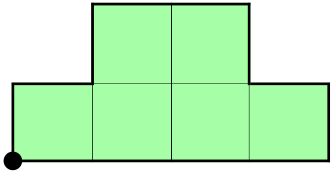

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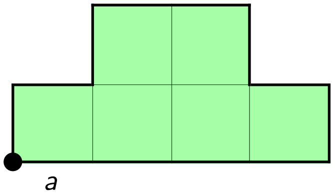

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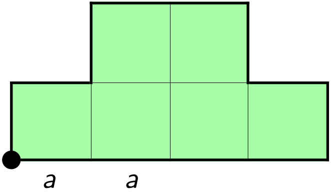

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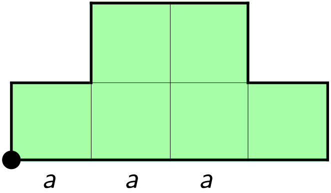

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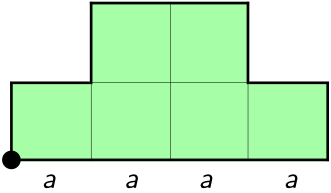

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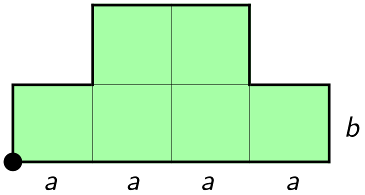

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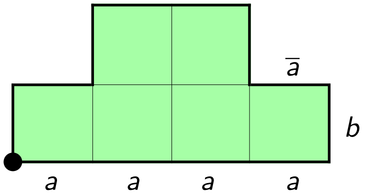

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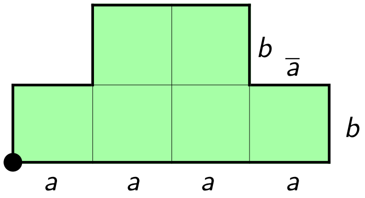

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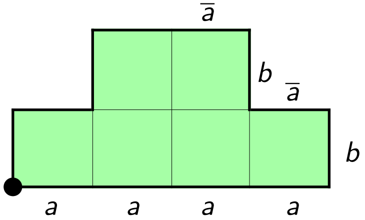

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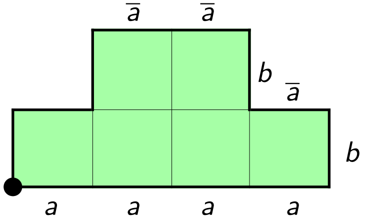

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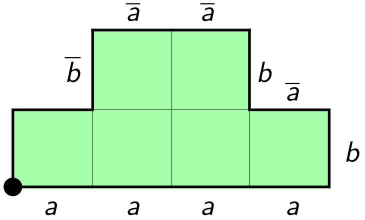

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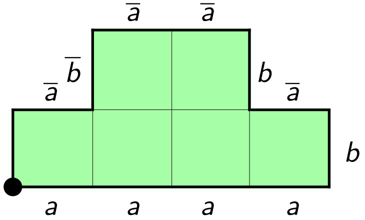

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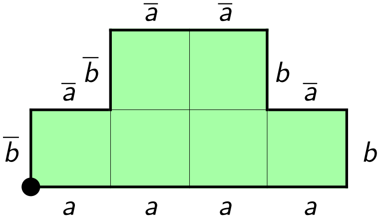

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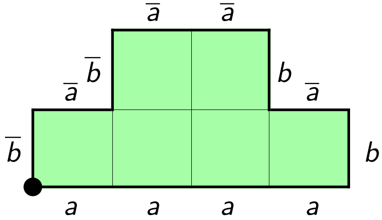

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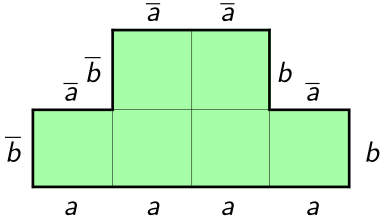

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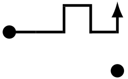

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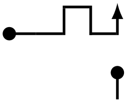

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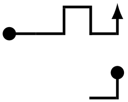

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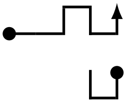

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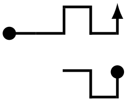

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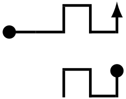

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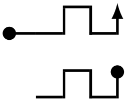

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Theorem (Beauquier and Nivat, 1991)

A polyomino p is exact if and only its boundary word $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$ for some $X, Y, Z \in \Sigma^*$.

Neighbouring

Definition

Two polyominoes p and q are simply neighbouring if

- *They are adjacent.*
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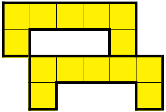

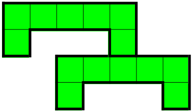

Triad

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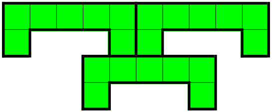

Surrounding

Definition

A surrounding of the polyomino p is an ordered sequence of translated copies $(p_0, p_1, \ldots, p_{k-1})$ such that for every i from 0 to k, the polyominoes p, p_i and p_{i+1} form a triad.

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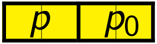

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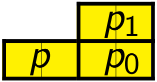

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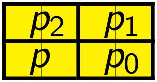

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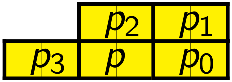

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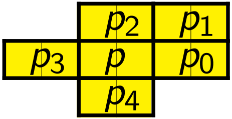

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A polyomino p is exact if and only if it admits a surrounding.

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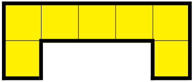

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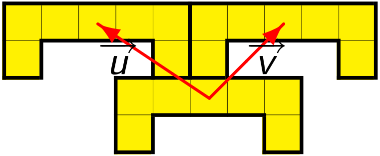

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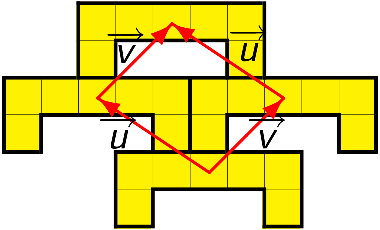
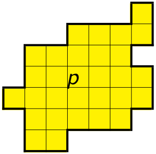


Example



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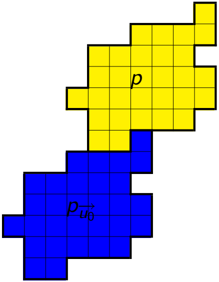

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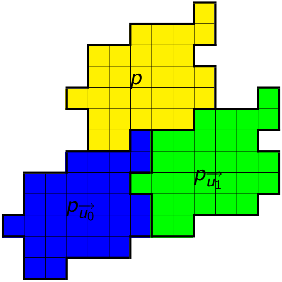

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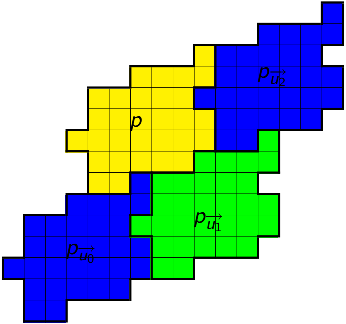
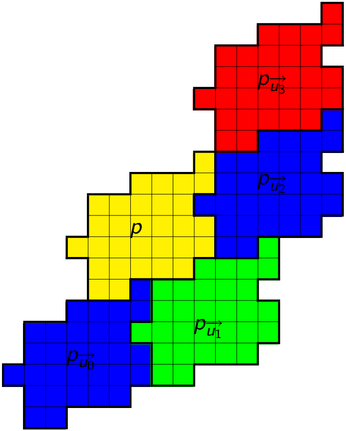

p
$p_{\overrightarrow{u_2}}$
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_0}}$

Example



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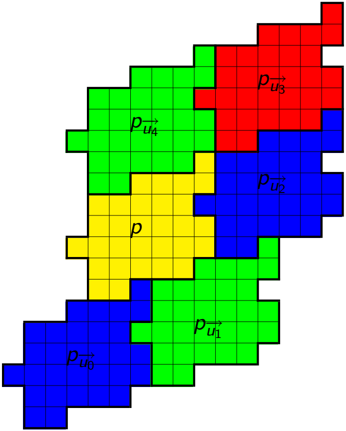

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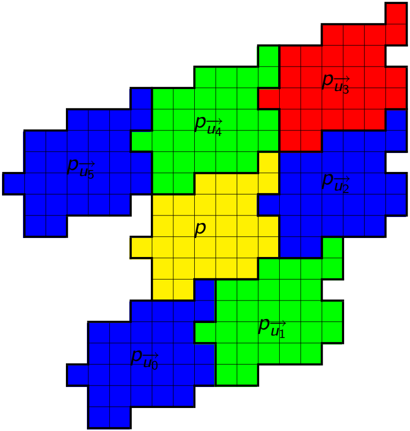
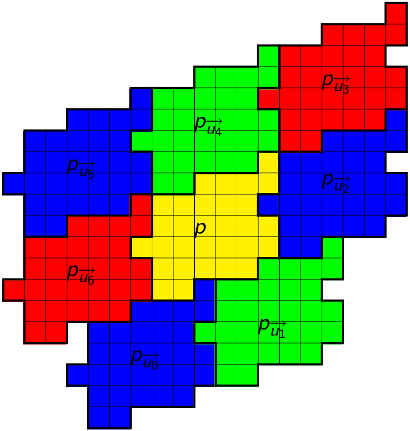

$p_{\overrightarrow{u_3}}$
$p_{\overrightarrow{u_4}}$
$p_{\overrightarrow{u_5}}$
$p_{\overrightarrow{u_2}}$
p
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_0}}$

Example


$p_{\overrightarrow{u_3}}$
$p_{\overrightarrow{u_4}}$
$p_{\overrightarrow{u_5}}$
$p_{\overrightarrow{u_2}}$
p
$p_{\overrightarrow{u_6}}$
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_0}}$

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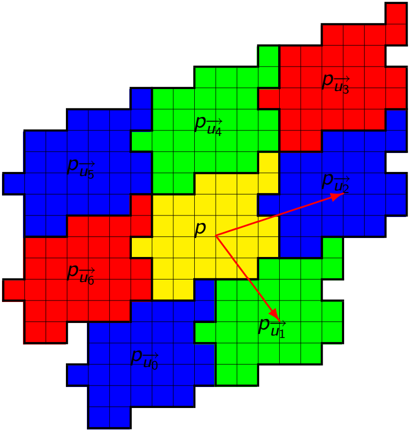

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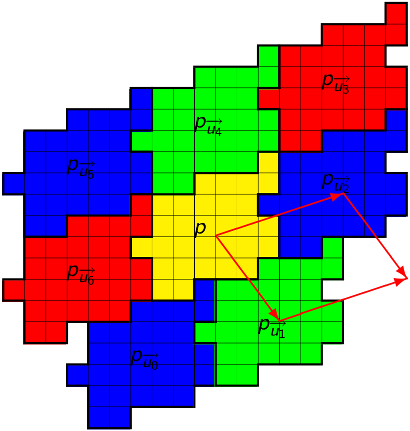
$P_{\overrightarrow{u_3}}$
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$P_{\overrightarrow{u_2}}$
P
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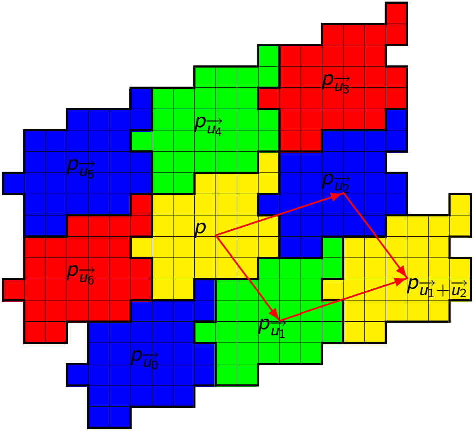
$p_{\vec{u_3}}$
$p_{\vec{u_4}}$
$p_{\vec{u_5}}$
$p_{\vec{u_2}}$
p
$p_{\vec{u_6}}$
$p_{\vec{u_1}+\vec{u_2}}$
$p_{\vec{u_1}}$
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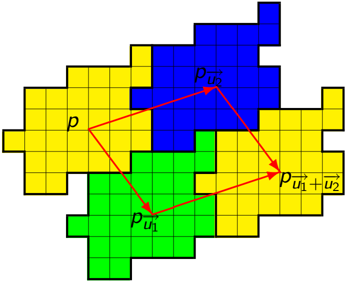
p
$p_{\overrightarrow{u_2}}$
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_1}+\overrightarrow{u_2}}$

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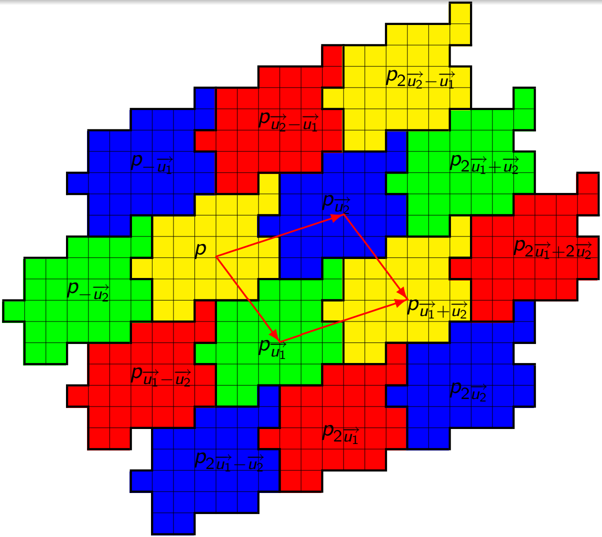
$P_{2\vec{u_2}-\vec{u_1}}$
$P_{\vec{u_2}-\vec{u_1}}$
$P_{-\vec{u_1}}$
$P_{2\vec{u_1}+\vec{u_2}}$
$P_{\vec{u_2}}$
P
$P_{2\vec{u_1}+2\vec{u_2}}$
$P_{-\vec{u_2}}$
$P_{\vec{u_1}+\vec{u_2}}$
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Surroundings and the factorization

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A polyomino p admits a surrounding if and only if its boundary word $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$ *for some* $X, Y, Z \in \Sigma^*$.

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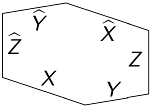

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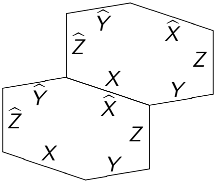

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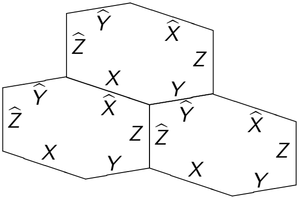

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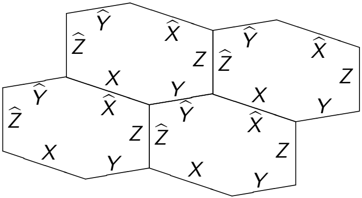

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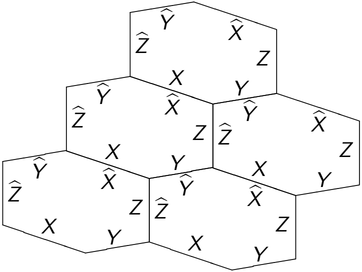

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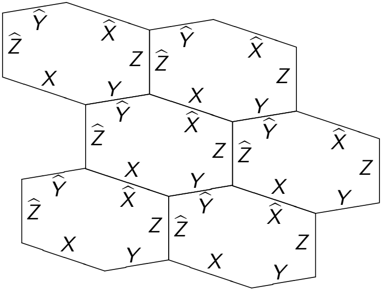

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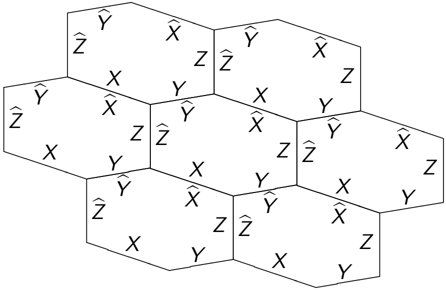

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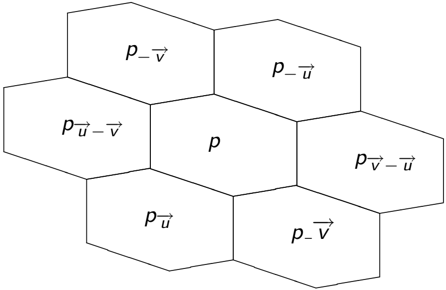

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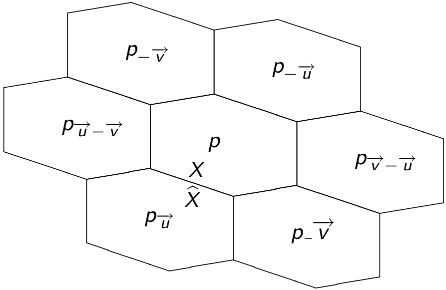

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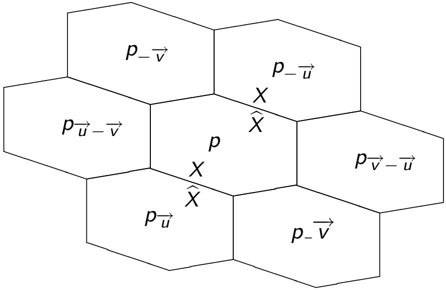

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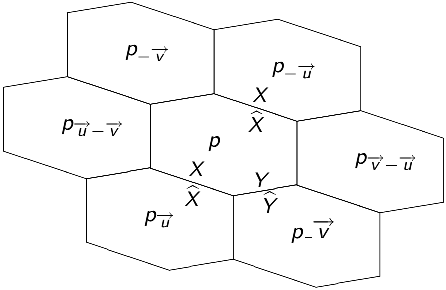

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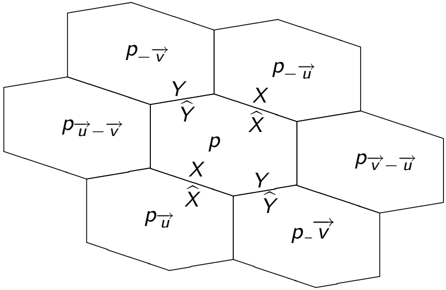

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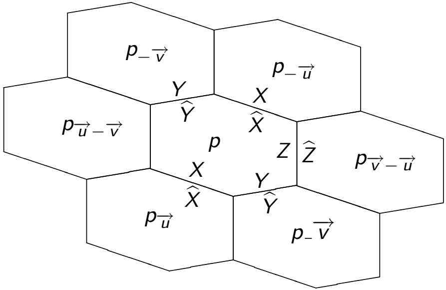

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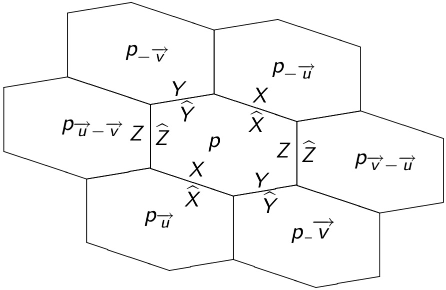

Pseudo-square and pseudo-hexagons

Definition

An exact polyomino p with Beauquier-Nivat factorization $XYZ\widehat{X}\widehat{Y}\widehat{Z}$ is called a pseudo-square if one of the factors X, Y, Z is the empty word. It is called a pseudo-hexagon otherwise.

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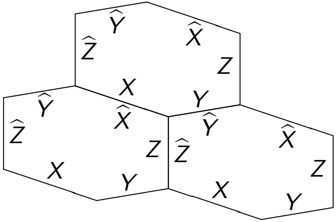

Pseudo-square

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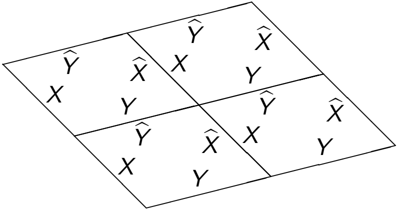

Complexity

Let n be the length of the word coding the boundary of a polyomino p.

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Theorem (Gambini and Vuillon, 2003)

There is an algorithm to test if a polyomino satisfies the Beauquier-Nivat characterization in $\mathcal{O}(n^2)$.

Outline

1 The tiling problem

2 Beauquier-Nivat characterization

3 A fast algorithm to detect exact polyominoes

Admissible factors

Definition

Let A be a factor of the word w coding a polyomino p. A is admissible if

- $w \equiv Ax\widehat{A}y$, *for* x, y *such that* $|x| = |y|$.
- A *is maximal, that is,* $\mathrm{first}(x) \neq \overline{\mathrm{last}(x)}$ *and* $\mathrm{first}(y) \neq \overline{\mathrm{last}(y)}$.

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Proposition

Let $\mathcal{A}$ be the set of all admissible factors overlapping a position α in w and $\widehat{\mathcal{A}}$ be the set of their respective homologous factors. Then, there is at least one position in w that is not covered by any element of $\mathcal{A} \cup \widehat{\mathcal{A}}$.

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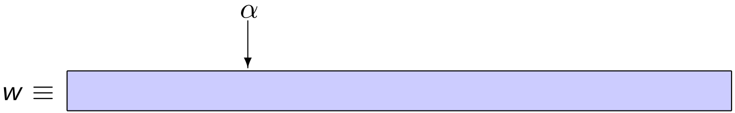

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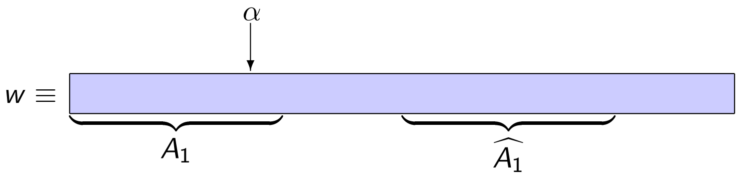

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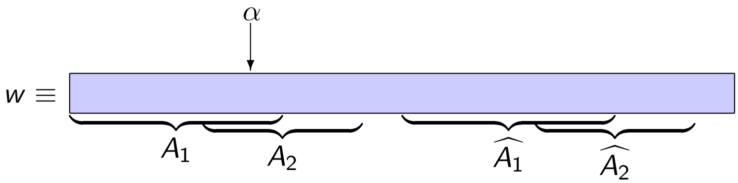

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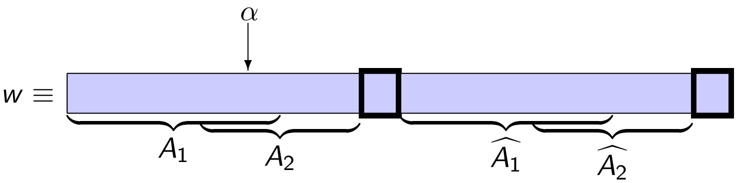

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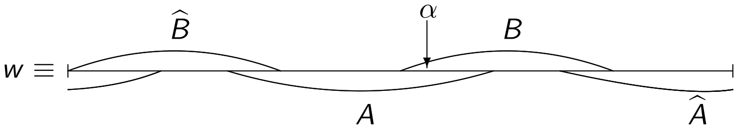

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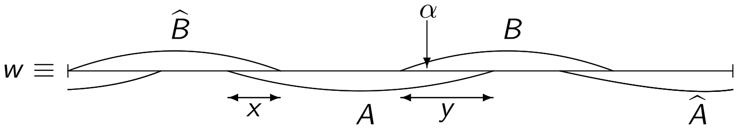

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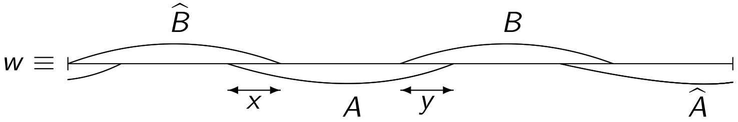

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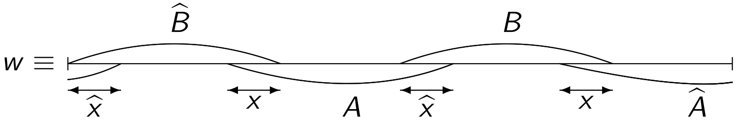

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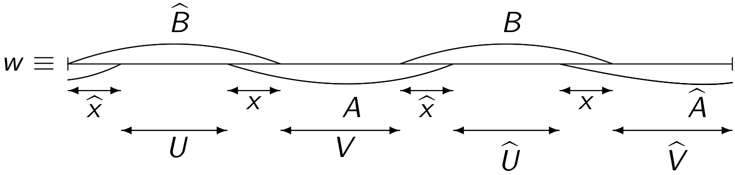

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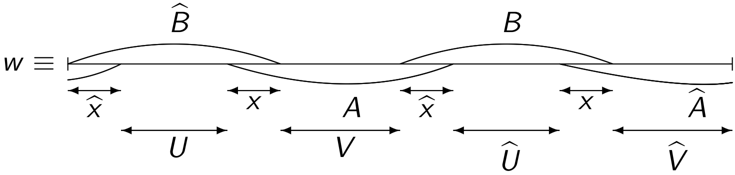

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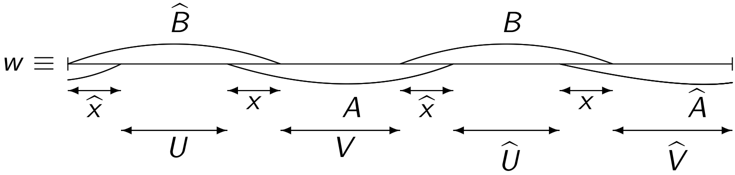

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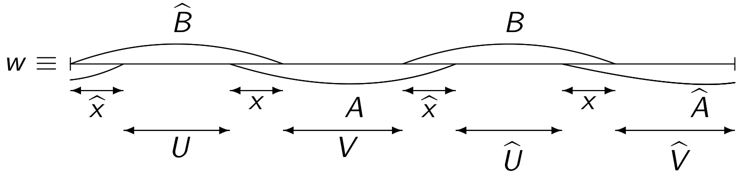

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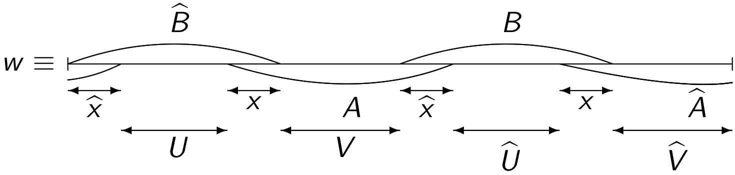

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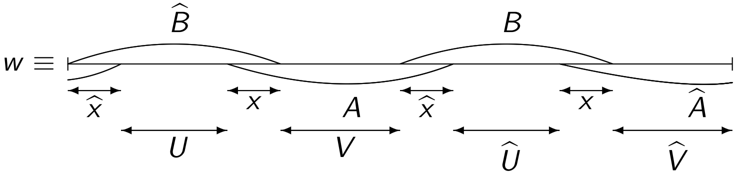

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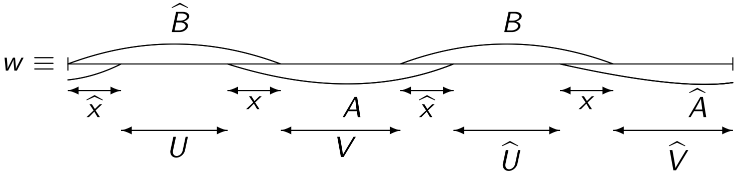

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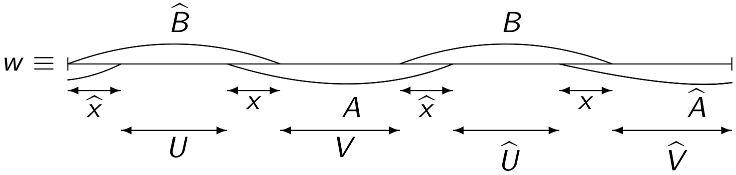

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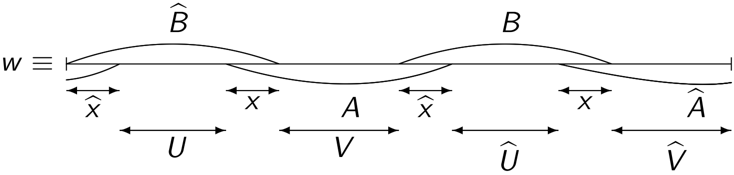

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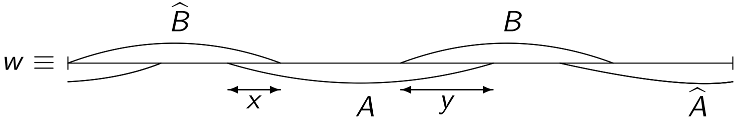

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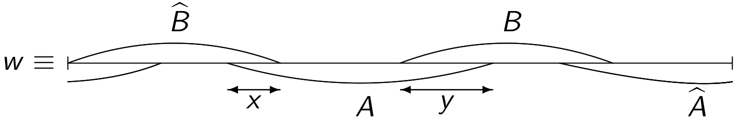

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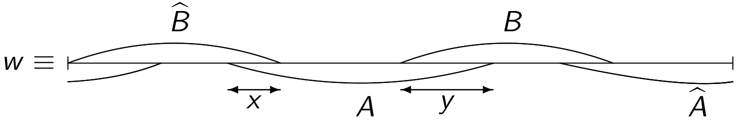

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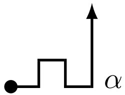

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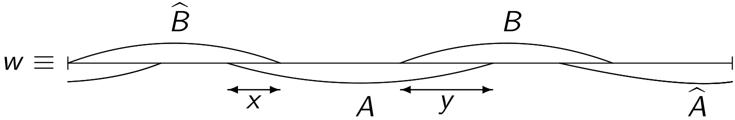

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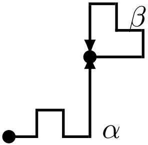

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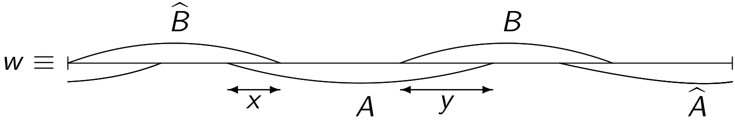

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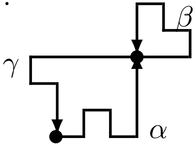

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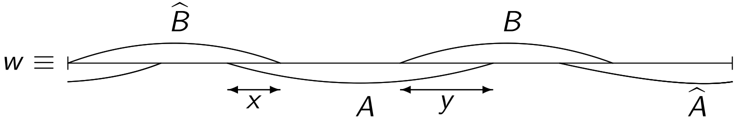

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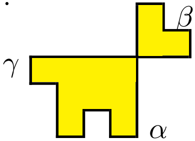

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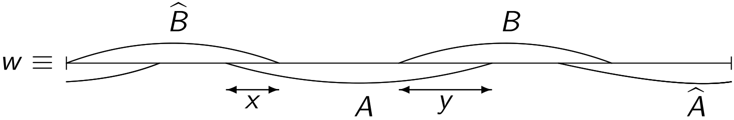

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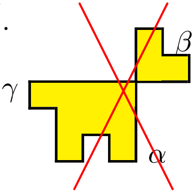

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  By contradiction, assume that X is not maximal, then $\mathrm{first}(YZ) = \overline{\mathrm{last}(YZ)}$.

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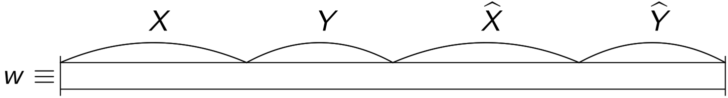

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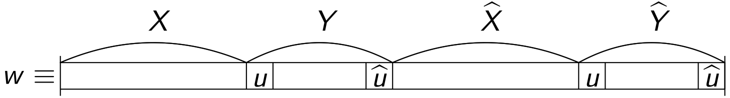

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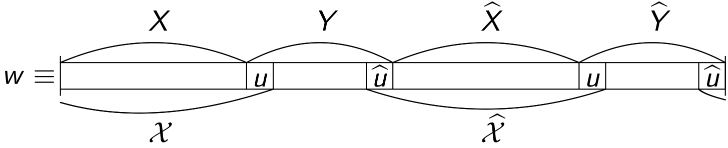

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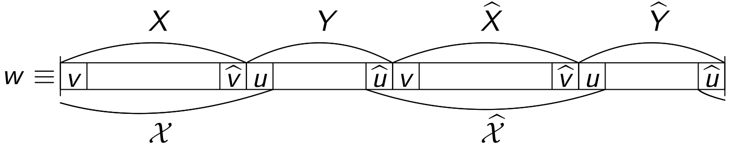

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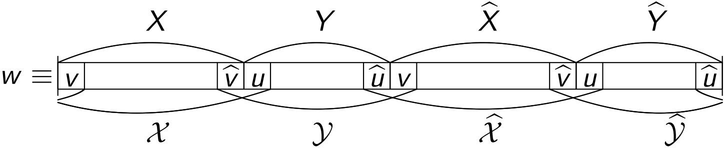

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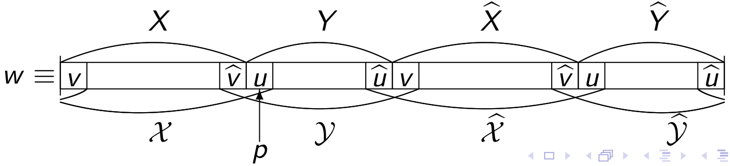

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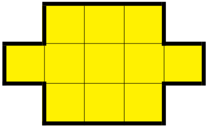

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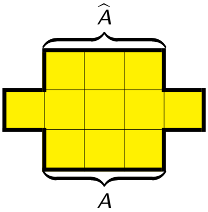

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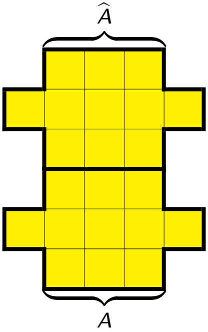

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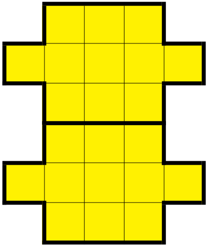

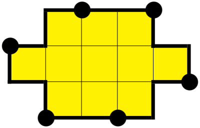

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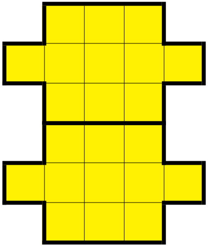

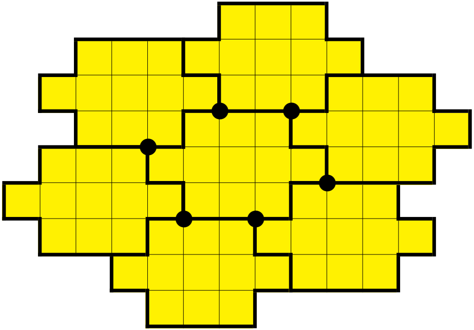

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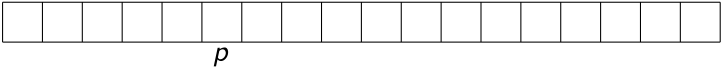
$w \equiv$ 

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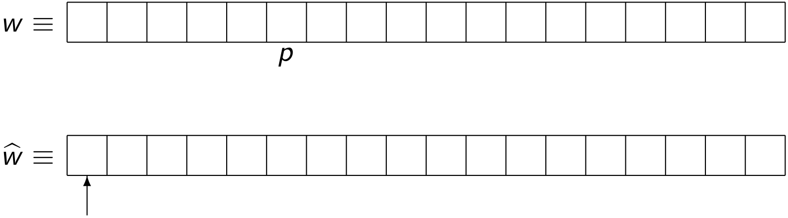

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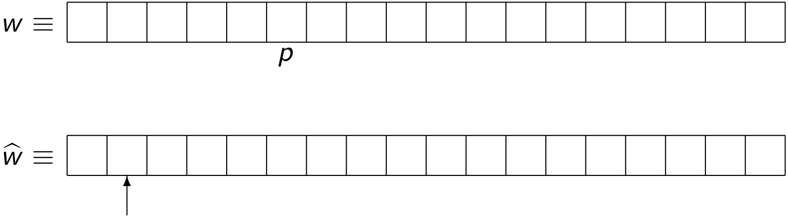

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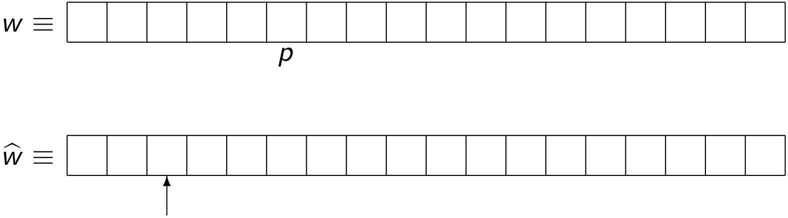

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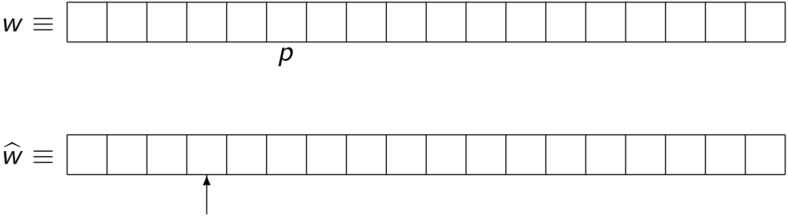

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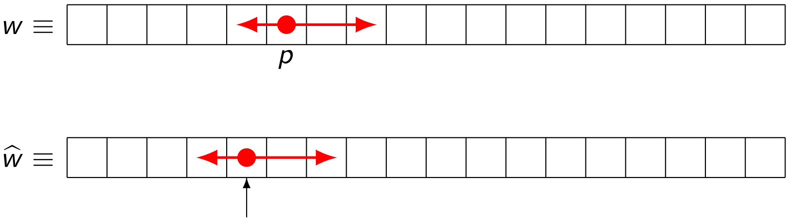

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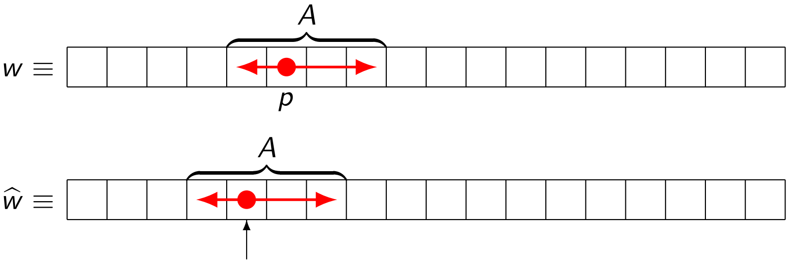

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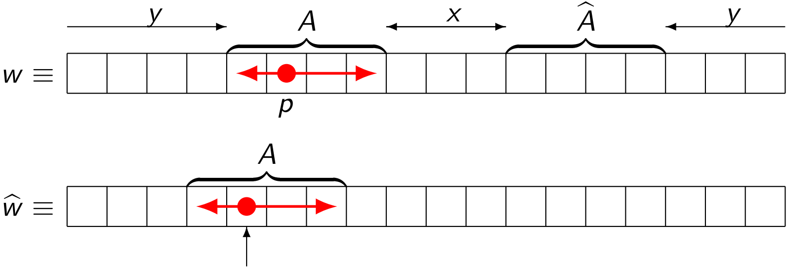

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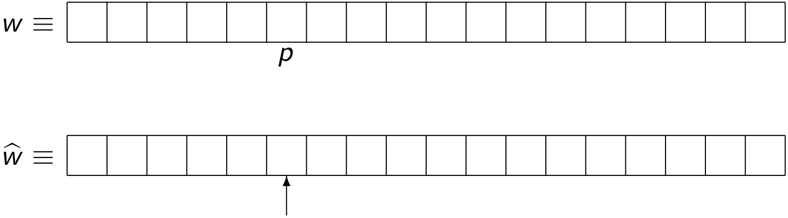

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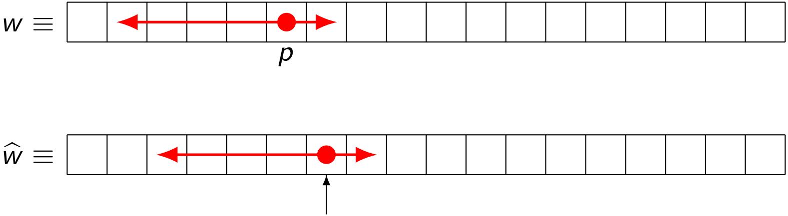

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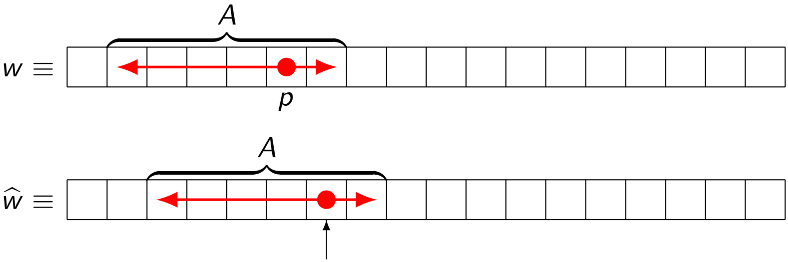

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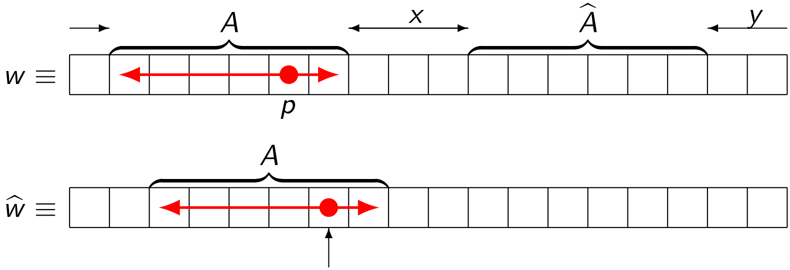

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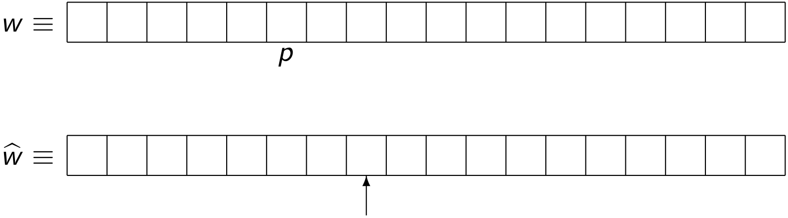

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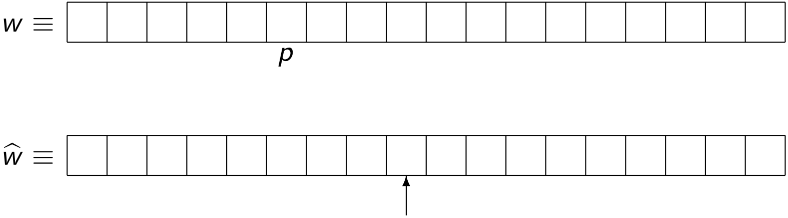

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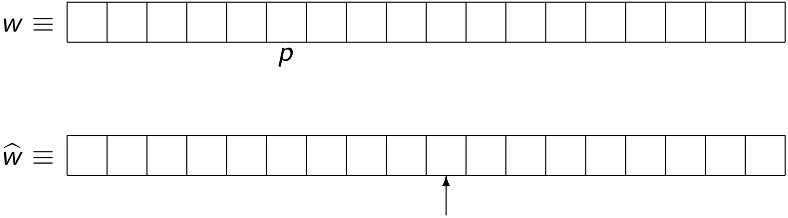

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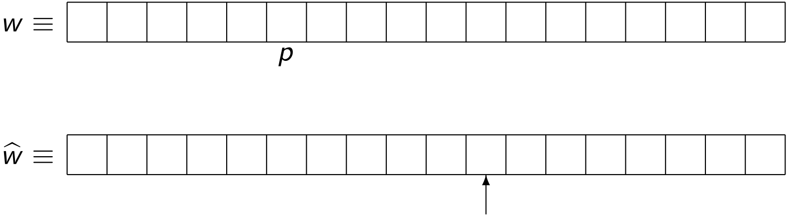

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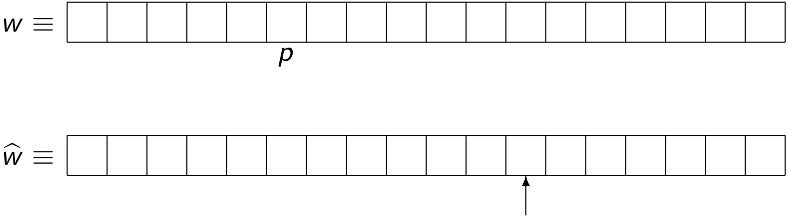

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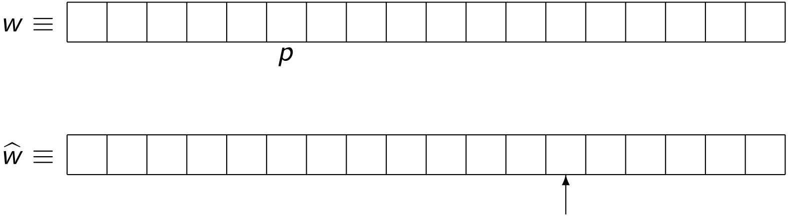

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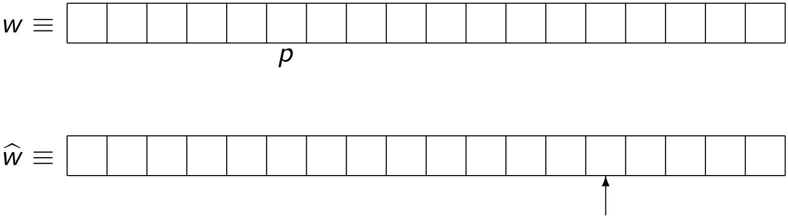

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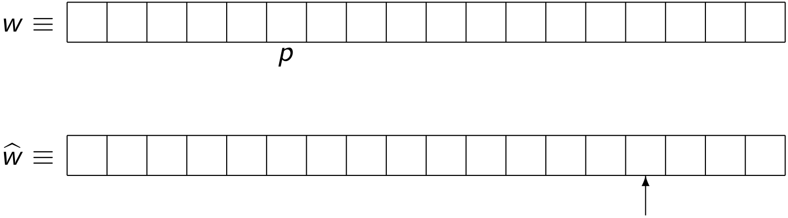

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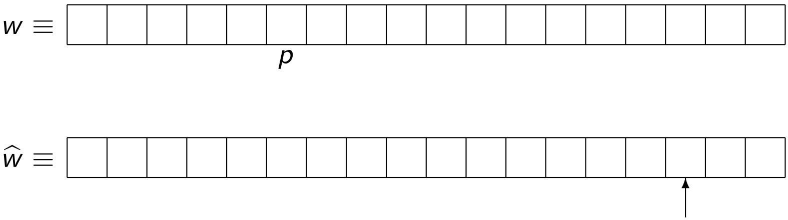

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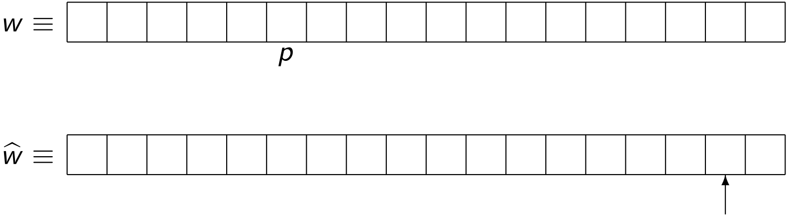

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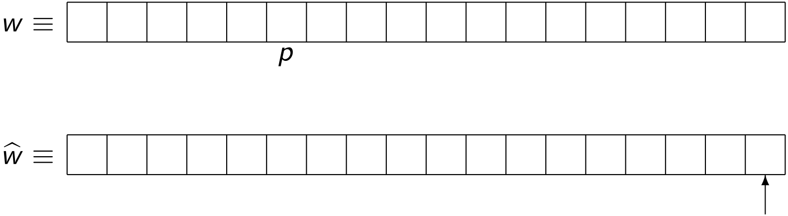

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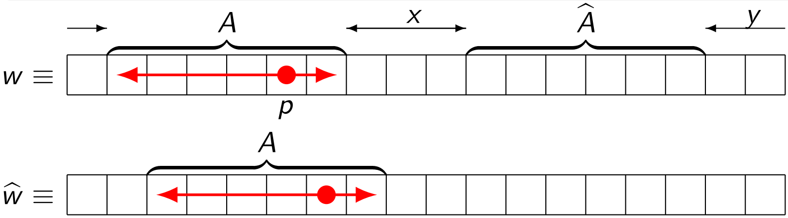

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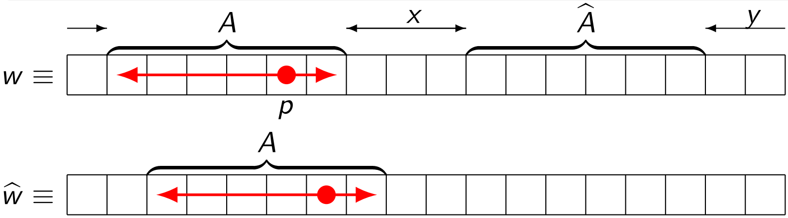

If $x = \widehat{y}$ then $w \equiv XY\widehat{X}\widehat{Y}$.

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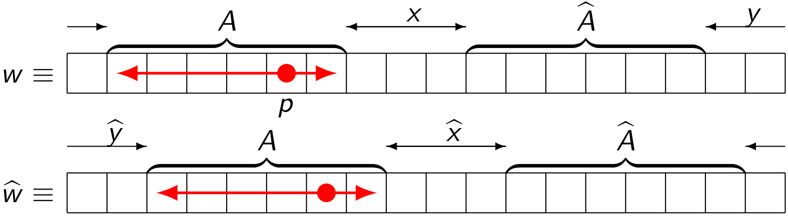

If $x = \widehat{y}$ then $w \equiv XY\widehat{X}\widehat{Y}$.
Since $w \equiv A\,x\,\widehat{A}\,y$ then $\widehat{w} \equiv \widehat{y}\,A\,\widehat{x}\,\widehat{A}$.

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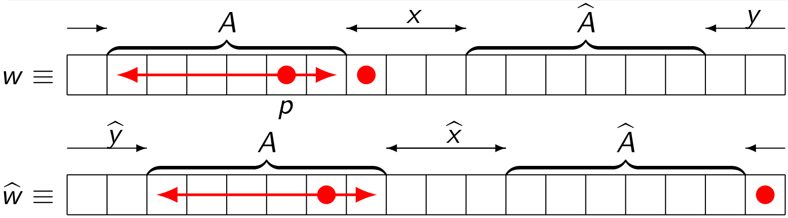

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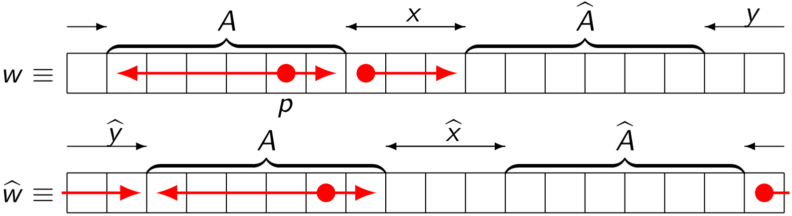

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Lemma

Let w be a k-square-free word coding a polyomino, and let α be a position in w. the number of admissible factors overlapping α in w is bounded by $4k + 2\log(n)$.

Detecting pseudo-hexagons

Theorem

Let w be a k-square-free word coding a polyomino, with $k \in \mathcal{O}(\sqrt{n})$. Determining if w codes a pseudo-hexagon is decidable in linear time.

Detecting pseudo-hexagons

Input : $w \in \Sigma^*$ coding a polyomino p.

Build L_1 the list of all admissible factors that overlap the position α.
$\beta :=$ (the position of the rightmost letter of w include in a factor of L_1) $+ 1$.
Build L_2 the list of all admissible factors that overlap the position β.
For all $X \in L_1$ **do**
  For all $Y \in L_2$ **do**
    If $w \equiv XYx\widehat{X}\widehat{Y}y$ **then**
      Compute i : the position of x in w.
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      If longest common extention$(w, \widehat{w}, i, j) = |x|$ **then**
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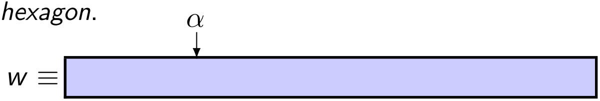

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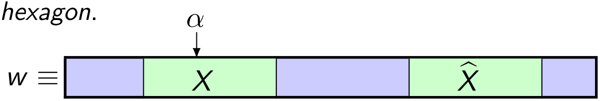

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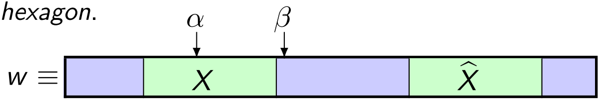

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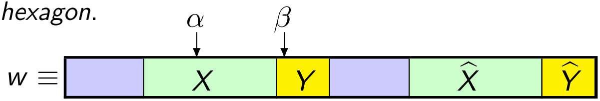

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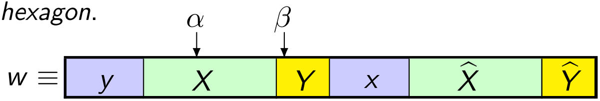

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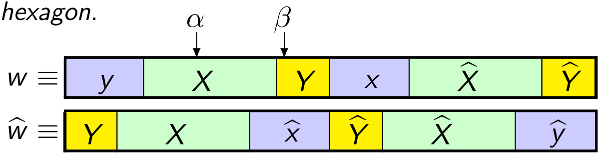

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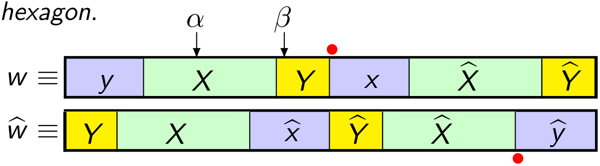

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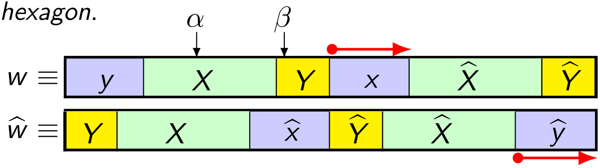

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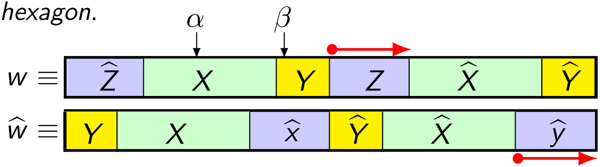

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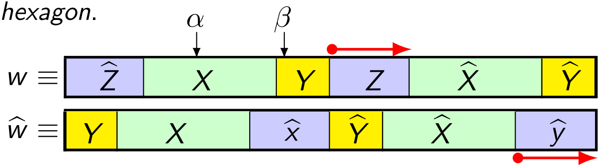

THANK YOU!