

Facet Connectedness of Discrete Hyperplanes with Zero Intercept: the General Case

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Abstract. A digital discrete hyperplane in \mathbb{Z}^d is defined by a normal vector \mathbf{v} , a shift μ , and a thickness ω . The set of thicknesses ω for which the hyperplane is connected is a right unbounded interval of \mathbb{R}^+ . Its lower bound, called the *connecting thickness* of \mathbf{v} with shift μ , may be computed by means of the fully subtractive algorithm. A careful study of the behaviour of this algorithm allows us to give exhaustive results about the connectedness of the hyperplane at the connecting thickness in the case $\mu = 0$. We show that it is connected if and only if the sequence of vectors computed by the algorithm reaches in finite time a specific set of vectors which has been shown to be Lebesgue negligible by Kraaikamp & Meester.

Keywords: discrete hyperplane, connectedness, connecting thickness, fully subtractive algorithm.

1 Preliminaries

In order to prevent any ambiguity, we denote by \mathbb{N}_0 the set of nonnegative integers ($\mathbb{N}_0 = \{0, 1, 2, \dots\}$), and by \mathbb{N}_1 the set of positive integers ($\mathbb{N}_1 = \{1, 2, \dots\}$). Given $d \in \mathbb{N}_1$, $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ denotes the canonical basis of \mathbb{R}^d . The usual scalar product on \mathbb{R}^d is denoted by $\langle \cdot, \cdot \rangle$. For any vector $\mathbf{v} \in \mathbb{R}^d$, $\|\mathbf{v}\|_1$ and $\|\mathbf{v}\|_\infty$ denote respectively the usual 1-norm and ∞ -norm of \mathbf{v} , which means $\|\mathbf{v}\|_1 = \sum_{i=1}^d |v_i|$, and $\|\mathbf{v}\|_\infty = \max_i |v_i|$. Given $\mathbf{v} \in \bigcup_{d \geq 1} \mathbb{R}^d$, we denote by $\#\mathbf{v}$ the dimension of the space \mathbf{v} belongs to, that is to say, $\mathbf{v} \in \mathbb{R}^{\#\mathbf{v}}$. Given $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ we denote by $\dim_{\mathbb{Q}}(v_1, \dots, v_d)$, or simply by $\dim_{\mathbb{Q}}(\mathbf{v})$ the dimension of $v_1 \mathbb{Q} + \dots + v_d \mathbb{Q}$ as a vector space over \mathbb{Q} . If $\dim_{\mathbb{Q}}(v_1, \dots, v_d) = 1$ then we denote by $\gcd(v_1, \dots, v_d)$, or simply by $\gcd(\mathbf{v})$, the greatest real number γ such that v_i/γ is an integer for all i . Two distinct points \mathbf{x} and \mathbf{y} in \mathbb{Z}^d are *facet-neighbours* (neighbours for short) if $\|\mathbf{x} - \mathbf{y}\|_1 = 1$ or equivalently if $\mathbf{x} - \mathbf{y} = \pm \mathbf{e}_i$ for some $i \in \{1, \dots, d\}$. This notion of neighbouring refers to the representation of a point in \mathbb{Z}^d as a voxel, i.e as a unit cube centred at the point. Two points are facet-neighbours if the voxels representing them share a facet. A *path* in \mathbb{Z}^d is a sequence $(\mathbf{x}^1, \dots, \mathbf{x}^n)$ such that \mathbf{x}^{i-1} and \mathbf{x}^i are neighbours

for all $i \in \{2, \dots, n\}$. A subset S of \mathbb{Z}^d is *connected* if it is not empty, and for all pairs of points \mathbf{x} and \mathbf{y} in S , there exists a path $(\mathbf{x}^1, \dots, \mathbf{x}^n)$ in S such that $\mathbf{x}^1 = \mathbf{x}$ and $\mathbf{x}^n = \mathbf{y}$.

Given a vector $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, and two real numbers μ and ω , the *arithmetic discrete hyperplane* with *normal vector* \mathbf{v} , *shift* μ , and *thickness* ω [1,11], denoted by $\mathbb{P}(\mathbf{v}, \mu, \omega)$, is the subset of \mathbb{Z}^d defined by

$$\mathbb{P}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^d \mid 0 \leq \langle \mathbf{v}, \mathbf{x} \rangle + \mu < \omega\}. \quad (1)$$

Given a vector $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and a shift $\mu \in \mathbb{R}$, we are interested in the set of values ω for which $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected. This set is known to be a right-unbounded interval of \mathbb{R}^+ [5]. Its lower bound is called the *connecting thickness of \mathbf{v} with shift μ* , and is denoted by $\Omega(\mathbf{v}, \mu)$. By definition, $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if $\omega > \Omega(\mathbf{v}, \mu)$, and disconnected if $\omega < \Omega(\mathbf{v}, \mu)$. The question that we address in this work is the connectedness of $\mathbb{P}(\mathbf{v}, \mu, \omega)$ at the critical thickness $\Omega(\mathbf{v}, \mu)$, i.e. whether $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected or not. In most cases, it is easily shown that the answer is negative. Only for a specific class of vectors, the answer was unknown up to now, although some partial results have been established [2,3]. We present here the general case when $\mu = 0$.

The problem of computing the connecting thickness has already been addressed several times [4,5,6,8,9,10]. This computation may be performed by means of the *fully subtractive algorithm* [12]. We recall it briefly in the next section, and by the way, we give some useful properties.

2 Computation of the connecting thickness

Let us start by giving some bounds on $\Omega(\mathbf{v}, \mu)$ which will be useful later.

Theorem 1 ([5]). *Let $d \geq 1$, $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, and $\mu, \omega \in \mathbb{R}$.*

- *If \mathbf{v} has exactly one non zero coordinate v_i , then $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if and only if $\omega > \mu \bmod |v_i| = \mu \bmod \gcd(\mathbf{v})$. Therefore, for all μ , we have $\Omega(\mathbf{v}, \mu) = \mu \bmod \gcd(\mathbf{v})$.*
 - *If \mathbf{v} has at least two non zero coordinates then let $\xi(\mathbf{v}) = \min\{|v_i| \mid v_i \neq 0\}$.*
 - *If $\omega \leq \|\mathbf{v}\|_\infty$ then $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is disconnected for all μ .*
 - *If $\omega \geq \|\mathbf{v}\|_\infty + \xi(\mathbf{v})$ then $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected for all μ .*
- Therefore, for all μ , we have $\|\mathbf{v}\|_\infty \leq \Omega(\mathbf{v}, \mu) \leq \|\mathbf{v}\|_\infty + \xi(\mathbf{v})$.*

The problem of computing the connecting thickness may first be simplified thanks to the following relation [5] which allows us to get rid of the shift μ in the computations:

$$\Omega(\mathbf{v}, \mu) = \begin{cases} \Omega(\mathbf{v}, 0) + (\mu \bmod \gcd(\mathbf{v})) & \text{if } \dim_{\mathbb{Q}}(\mathbf{v}) = 1; \\ \Omega(\mathbf{v}, 0) & \text{if } \dim_{\mathbb{Q}}(\mathbf{v}) \geq 2. \end{cases}$$

We are then left to compute $\Omega(\mathbf{v}, 0)$ that we simply denote by $\Omega(\mathbf{v})$. For convenience, we shall usually write $\Omega(v_1, \dots, v_d)$ instead of $\Omega((v_1, \dots, v_d))$. The problem may be further simplified thanks to the observation that for any permutation

σ of $\{1, \dots, d\}$, $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if and only if $\mathbb{P}((|v_{\sigma(1)}|, \dots, |v_{\sigma(d)}|), \mu, \omega)$ is connected [5]. We may therefore assume that \mathbf{v} belongs to $(\mathbb{R}^+)^d \setminus \{\mathbf{0}\}$, and that its coordinates are suitably ordered. In the sequel, we denote by \mathcal{O}_d^+ the set of vectors $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ such that $0 \leq v_1 \leq \dots \leq v_d$. Finally, if $\mathbf{v} \in \mathcal{O}_d^+$ and $v_1 = 0$, then $\mathbb{P}((0, v_2, \dots, v_d), \mu, \omega)$ is connected if and only if $\mathbb{P}((v_2, \dots, v_d), \mu, \omega)$ is connected.

The following theorem is the key to the computation of $\Omega(\mathbf{v})$. It appears under various forms and in various contexts in the literature.

Theorem 2 ([5,6,8,9,10]). *Let $d \geq 2$, $\mathbf{v} \in \mathcal{O}_d^+$, and $\mathbf{v}' = (v_1, v_2 - v_1, \dots, v_d - v_1)$. For all $\mu, \omega \in \mathbb{R}$, $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if and only if $\mathbb{P}(\mathbf{v}', \mu, \omega - v_1)$ is connected. Therefore, $\Omega(\mathbf{v}, \mu) = v_1 + \Omega(\mathbf{v}', \mu)$.*

We define π and σ as:

$$\begin{aligned} \pi : \bigcup_{d \geq 2} \mathbb{R}^d &\rightarrow \bigcup_{d \geq 2} \mathbb{R}^{d-1} \\ (v_1, v_2, \dots, v_d) &\mapsto (v_2, \dots, v_d); \\ \\ \sigma : \bigcup_{d \geq 2} \mathbb{R}^d &\rightarrow \bigcup_{d \geq 2} \mathbb{R}^d \\ (v_1, v_2, \dots, v_d) &\mapsto (v_1, v_2 - v_1, \dots, v_d - v_1). \end{aligned}$$

Thanks to Th. 2 and to the preceding remarks, given $\mathbf{v} \in \mathcal{O}_d^+$, we may compute $\Omega(\mathbf{v})$ recursively as follows:

$$\Omega(\mathbf{v}) = \begin{cases} 0 & \text{if } \#\mathbf{v} = 1, \\ \Omega(\pi(\mathbf{v})) & \text{if } \#\mathbf{v} \geq 2 \text{ and } v_1 = 0, \\ v_1 + \Omega(\text{sort}(\sigma(\mathbf{v}))) & \text{if } \#\mathbf{v} \geq 2 \text{ and } v_1 > 0, \end{cases}$$

where ‘sort’ orders the coordinates of its argument in non decreasing order.

The algorithm deduced from these equations is known as the *Ordered Fully Subtractive algorithm (OFS)* [12]. In effect, OFS computes a possibly infinite sequence of pairs $(\mathbf{v}^n, \Omega^n)_{n \geq 1}$ defined by:

$$\boxed{\begin{aligned} (\mathbf{v}^1, \Omega^1) &= (\text{sort}(\mathbf{v}), 0) \text{ and for all } n \geq 1 \text{ such that } \#\mathbf{v}^n \geq 2, \\ (\mathbf{v}^{n+1}, \Omega^{n+1}) &= \begin{cases} (\pi(\mathbf{v}^n), \Omega^n) & \text{if } v_1^n = 0; \\ (\text{sort}(\sigma(\mathbf{v}^n)), \Omega^n + v_1^n) & \text{if } v_1^n > 0. \end{cases} \end{aligned}}$$

If $\#\mathbf{v}^n = 1$ for some n , then OFS actually terminates, and the sequence is finite. This sequence has the following properties. For all $\omega, \mu \in \mathbb{R}$ and all $n \geq 1$ such that (\mathbf{v}^n, Ω^n) is defined:

- $\Omega^n = \sum_{i=1}^{n-1} v_1^i$;
- $\Omega(\mathbf{v}, \mu) = \Omega^n + \Omega(\mathbf{v}^n, \mu)$;
- $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if and only if $\mathbb{P}(\mathbf{v}^n, \mu, \omega - \Omega^n)$ is connected;
- $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected if and only if $\mathbb{P}(\mathbf{v}^n, \mu, \Omega(\mathbf{v}^n, \mu))$ is connected.

At each step of the algorithm, either \mathbf{v} decreases componentwise, or the number of coordinates of \mathbf{v} decreases, which may happen only finitely many times. Also, Ω^n is increasing and bounded by $\Omega(\mathbf{v})$. OFS is therefore convergent in the sense that \mathbf{v}^n and Ω^n tend towards limits \mathbf{v}^∞ and Ω^∞ . However, it terminates if and only if $\dim_{\mathbb{Q}}(v_1, \dots, v_d) = 1$.

If OFS terminates, which means that at some step n_0 , we have $\#\mathbf{v}^{n_0} = 1$, then $\Omega(\mathbf{v}) = \Omega^{n_0}$, and for all $\mu \in \mathbb{R}$,

$$\Omega(\mathbf{v}, \mu) = \Omega^{n_0} + (\mu \bmod \gcd(\mathbf{v})). \quad (2)$$

If OFS does not terminate, then for all $\mu \in \mathbb{R}$, we have

$$\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{v}) = \Omega^\infty + \|\mathbf{v}^\infty\|_\infty. \quad (3)$$

Indeed, for all $n \geq 1$, we have $\Omega(\mathbf{v}, \mu) = \Omega^n + \Omega(\mathbf{v}^n, \mu)$, and, by Th. 1, we have $\|\mathbf{v}^n\|_\infty \leq \Omega(\mathbf{v}^n, \mu) \leq \|\mathbf{v}^n\|_\infty + \xi(\mathbf{v}^n)$. Since $\lim_{n \rightarrow \infty} \xi(\mathbf{v}^n) = 0$, we get $\Omega(\mathbf{v}, \mu) = \lim_{n \rightarrow \infty} (\Omega^n + \Omega(\mathbf{v}^n, \mu)) = \Omega^\infty + \|\mathbf{v}^\infty\|_\infty$. In this case, there must exist n_0 such that $\#\mathbf{v}^{n_0} \geq 2$, and $v_1^{n_0} > 0$ for all $n \geq n_0$. Let $d_\infty = \#\mathbf{v}^{n_0}$. Note that $d_\infty \geq 2$. Then for all $n \geq n_0$, we have $\|\mathbf{v}^{n+1}\|_1 = \|\mathbf{v}^n\|_1 - (d_\infty - 1)v_1^n$ so that

$$\Omega^\infty = \Omega^{n_0} + \sum_{n=n_0}^{\infty} v_1^n = \Omega^{n_0} + \sum_{n=n_0}^{\infty} \frac{\|\mathbf{v}^n\|_1 - \|\mathbf{v}^{n+1}\|_1}{d_\infty - 1} = \Omega^{n_0} + \frac{\|\mathbf{v}^{n_0}\|_1 - \|\mathbf{v}^\infty\|_1}{d_\infty - 1}.$$

For all $\mu \in \mathbb{R}$, we get

$$\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{v}) = \Omega^{n_0} + \frac{\|\mathbf{v}^{n_0}\|_1 - \|\mathbf{v}^\infty\|_1}{d_\infty - 1} + \|\mathbf{v}^\infty\|_\infty. \quad (4)$$

In particular, if $v_1^n > 0$ for all n , then $n_0 = 1$ and $d_\infty = d$, so that

$$\Omega(\mathbf{v}) = \frac{\|\mathbf{v}\|_1 - \|\mathbf{v}^\infty\|_1}{d - 1} + \|\mathbf{v}^\infty\|_\infty. \quad (5)$$

Theorem 3 ([9,10]). *Let $d \geq 2$ and $\mathbf{v} \in \mathcal{O}_d^+$. If $v_1^n > 0$ for all n , then*

$$\lim_{n \rightarrow \infty} \mathbf{v}^n = \mathbf{0} \text{ if and only if } \|\mathbf{v}^n\|_\infty \leq \|\mathbf{v}^n\|_1 / (d - 1) \text{ for all } n.$$

According to this theorem, if $\|\mathbf{v}^n\|_\infty \leq \|\mathbf{v}^n\|_1 / (d_\infty - 1)$ for all $n \geq n_0$, then Eq. (4) becomes $\Omega(\mathbf{v}) = \Omega^{n_0} + (\|\mathbf{v}^{n_0}\|_1) / (d_\infty - 1)$. In particular, if $n_0 = 1$, we get

$$\Omega(\mathbf{v}) = \frac{\|\mathbf{v}\|_1}{d - 1}. \quad (6)$$

3 Connectedness at the connecting thickness

By definition of $\Omega(\mathbf{v}, \mu)$, $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is disconnected for all $\omega < \Omega(\mathbf{v}, \mu)$, and connected for all $\omega > \Omega(\mathbf{v}, \mu)$. The question we want to answer now is whether $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected. This question has an easy answer when OFS terminates, which means when $\dim_{\mathbb{Q}}(\mathbf{v}) = 1$.

Theorem 4. *If $\dim_{\mathbb{Q}}(\mathbf{v}) = 1$ then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is disconnected for all $\mu \in \mathbb{R}$.*

Proof. When $\dim_{\mathbb{Q}}(\mathbf{v}) = 1$, $\gcd(\mathbf{v}^n)$ is obviously an invariant of OFS. Therefore, when the halting condition is reached, we have $\mathbf{v}^n = \gamma \mathbf{e}_1$ where $\gamma = \gcd(v_1, \dots, v_d)$. Then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected if and only if $\mathbb{P}(\mathbf{v}^n, \mu, \Omega(\mathbf{v}^n, \mu))$ is connected. But $\Omega(\mathbf{v}^n, \mu) = \mu \bmod \gamma$, and $\mathbb{P}(\mathbf{v}^n, \mu, \Omega(\mathbf{v}^n, \mu)) = \{x \in \mathbb{Z} \mid 0 \leq \gamma x + \mu < \mu \bmod \gamma\}$. This set is empty, hence disconnected. \square

If OFS does not terminate, then $\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{v})$, and after some step n_0 , no coordinate of the vector vanishes anymore, meaning that $v_1^n > 0$ for all $n \geq n_0$. Then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is connected if and only if $\mathbb{P}(\mathbf{v}^{n_0}, \mu, \Omega(\mathbf{v}^{n_0}))$ is connected. Therefore, we shall now study the case of vectors such that $v_1^n > 0$ for all n .

Theorem 5. *If $v_1^n > 0$ for all n , and $\|\mathbf{v}^n\|_{\infty} \geq \|\mathbf{v}^n\|_1 / (d - 1)$ for some n , then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is disconnected for all $\mu \in \mathbb{R}$.*

The following lemma will be useful for the proof of this theorem.

Lemma 6. *Let $d \geq 3$ and $\mathbf{v} \in \mathcal{O}_d^+$. If there exists $r \in \{2, \dots, d - 1\}$ such that $\dim_{\mathbb{Q}}(v_1, \dots, v_r) \geq 2$, and $v_{r+1} \geq \Omega(v_1, \dots, v_r)$ then $\Omega(\mathbf{v}) = v_d$.*

Proof. If $v_1 = 0$ then we have $r \geq 3$ because $\dim_{\mathbb{Q}}(v_1, \dots, v_r) \geq 2$, and the conditions of the theorem still hold for (v_2, \dots, v_d) , taking $d' = d - 1$ and $r' = r - 1$. We may therefore assume, without loss of generality, that $v_1 > 0$. Then

$$\Omega(\mathbf{v}) = v_1 + \Omega(v_1, v_2 - v_1, \dots, v_r - v_1, v_{r+1} - v_1, \dots, v_d - v_1),$$

and

$$\Omega(v_1, \dots, v_r) = v_1 + \Omega(v_1, v_2 - v_1, \dots, v_r - v_1).$$

Let $\mathbf{v}' = (v_1, v_2 - v_1, \dots, v_d - v_1)$ and $\mathbf{v}'' = \text{sort}(\mathbf{v}')$. We have $v'_r > 0$ since otherwise we would have $v_1 = \dots = v_r$, hence $\dim_{\mathbb{Q}}(v_1, \dots, v_r) = 1$. We also have $\dim_{\mathbb{Q}}(v'_1, \dots, v'_r) = \dim_{\mathbb{Q}}(v_1, \dots, v_r) \geq 2$. Hence, (v'_1, \dots, v'_r) has at least two non zero coordinates. By hypothesis, we have $v'_{r+1} = v_{r+1} - v_1 \geq \Omega(v_1, v_2, \dots, v_r) - v_1 = \Omega(v_1, v_2 - v_1, \dots, v_r - v_1) = \Omega(v'_1, v'_2, \dots, v'_r)$, which, by Th. 1, implies $v'_{r+1} \geq \|(v'_1, \dots, v'_r)\|_{\infty}$. Hence, $(v''_1, \dots, v''_r) = \text{sort}(v'_1, \dots, v'_r)$, and $(v''_{r+1}, \dots, v''_d) = (v'_{r+1}, \dots, v'_d) = (v_{r+1} - v_1, \dots, v_d - v_1)$. Thus, \mathbf{v}'' still satisfies the conditions of the theorem. Furthermore, when we apply OFS to \mathbf{v} , for all n , we have $\|\mathbf{v}^n\|_{\infty} = v_{\# \mathbf{v}^n}^n = v_d - \Omega^n$. Therefore, $\Omega(\mathbf{v}) = \Omega^{\infty} + \|\mathbf{v}^{\infty}\|_{\infty} = \Omega^{\infty} + (v_d - \Omega^{\infty}) = v_d$. \square

Note to the reviewer:
full proof given
in the appendix.

Sketch of proof of Th. 5. For all $n \geq 1$ and all $r \in \{2, \dots, d\}$, we consider $K_r^n = (r - 1)v_r^n - (v_1^n + \dots + v_r^n)$. We check that, as long as $v_1^n > 0$, K_r^n is always increasing, both in r and in n . We define r_0 as the smallest index r such that $K_r^n \geq 0$ for some n , and n_0 as the smallest index n such that $K_{r_0}^n \geq 0$. Since $K_2^n = -v_1^n < 0$, we have $r_0 \geq 3$.

Then we prove that $\dim_{\mathbb{Q}}(v_1^{n_0}, \dots, v_{r_0-1}^{n_0}) \geq 2$, and that for all n , we have $\Omega(v_1^{n_0}, \dots, v_{r_0-1}^{n_0}) = (v_1^{n_0} + \dots + v_{r_0-1}^{n_0}) / (r_0 - 2) \leq v_{r_0}^{n_0}$. By Lemma. 6, we get $\Omega(\mathbf{v}^{n_0}) = v_d^{n_0} = \|\mathbf{v}^{n_0}\|_{\infty}$. Then, by Th. 1, $\mathbb{P}(\mathbf{v}^{n_0}, \mu, \Omega(\mathbf{v}^{n_0}))$, hence $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$, is disconnected for all $\mu \in \mathbb{R}$. \square

We are now left to consider the case where $v_1^n > 0$ and $\|\mathbf{v}^n\|_1 > (d-1)\|\mathbf{v}^n\|_\infty$ for all n . That is to say, where $v_1^n > 0$ for all n but \mathbf{v} does not satisfy the second condition of Th. 5. For $d \geq 2$, We note \mathcal{K}_d the set of vectors in $(\mathbb{R}^+)^d$ having this property. Kraaikamp & Meester [9] have shown that \mathcal{K}_d is Lebesgue negligible for all $d \geq 3$. If $\mathbf{v} \in \mathcal{K}_d$ then, by Th. 3, $\mathbf{v}^\infty = 0$ so that $\Omega(\mathbf{v}) = \|\mathbf{v}\|_1/(d-1)$. From what precedes, it is the only case where $v_1^n > 0$ for all n , and $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ could possibly be connected. Some partial results have already been published in the literature.

A first result was obtained in [2] for the case where $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$, and α is the inverse of the Tribonacci number, which means $\alpha^3 + \alpha^2 + \alpha - 1 = 0$. We have $\mathbf{v} \in \mathcal{K}_3$, and it has been shown that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected, and $\mathbb{P}(\mathbf{v}, \Omega(\mathbf{v}), \Omega(\mathbf{v}))$ is disconnected.

For the case $d = 3$, it has been shown in [3] that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected for all vectors in \mathcal{K}_3 . The proof relies on techniques from the field of substitutions on planes, and seems difficult to extend to higher dimensions.

In the sequel, we shall address the general case, and prove that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected for all $\mathbf{v} \in \cup_{d \geq 2} \mathcal{K}_d$. Note that in the case $d = 2$, the condition of Kraaikamp & Meester becomes $v_2^n < v_1^n + v_2^n$ which always holds if $v_1^n > 0$ for all n , i.e. if $\dim_{\mathbb{Q}}(v_1, v_2) = 2$. In this case, we have $\Omega(v_1, v_2) = v_1 + v_2 = \|\mathbf{v}\|_1$. By Th. 1, $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is connected for all $\mu \in \mathbb{R}$.

4 Main connectedness result

In order to establish the main result of this work, we need to study carefully the behaviour of the fully subtractive algorithm. To do so, we need to consider an unordered version of this algorithm, which means that the coordinates of the vector are not ordered anymore. This works as follows: as long as $\#\mathbf{v} \geq 2$, if some coordinate is zero, it is erased, otherwise, a minimal coordinate is subtracted from all other ones. We call this algorithm UFS for *Unordered Fully Subtractive*.

For all $k \geq 1$, we define σ_k and π_k as:

$$\begin{aligned} \sigma_k : \bigcup_{d \geq k} \mathbb{R}^d &\rightarrow \bigcup_{d \geq k} \mathbb{R}^d \\ (v_1, \dots, \mathbf{v}_k, \dots, v_d) &\mapsto (v_1 - v_k, \dots, v_{k-1} - v_k, \mathbf{v}_k, v_{k+1} - v_k, \dots, v_d - v_k); \end{aligned}$$

$$\begin{aligned} \pi_k : \bigcup_{d \geq k} \mathbb{R}^d &\rightarrow \bigcup_{d \geq k} \mathbb{R}^{d-1} \\ (v_1, \dots, \mathbf{v}_k, \dots, v_d) &\mapsto (v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_d). \end{aligned}$$

UFS computes a possibly infinite sequence of pairs $(\mathbf{v}^n, \Omega^n)_{n \geq 1}$ defined by:

$$\begin{aligned} (\mathbf{v}^1, \Omega^1) &= (\mathbf{v}, 0), \text{ and for all } n \geq 1 \text{ such that } \#\mathbf{v}^n \geq 2, \\ (\mathbf{v}^{n+1}, \Omega^{n+1}) &= \begin{cases} (\pi_{i_0}(\mathbf{v}^n), \Omega^n) & \text{if } v_{i_0}^n = 0; \\ (\sigma_{i_0}(\mathbf{v}^n), \Omega^n + v_{i_0}^n) & \text{if } v_{i_0}^n = \min_i v_i^n > 0. \end{cases} \end{aligned}$$

Note that UFS is not deterministic since several coordinates of \mathbf{v}^n could be minimal. However, OFS and UFS generate the same sequence $(\Omega^n)_{n \geq 1}$ and if $(\mathbf{v}^n)_{n \geq 1}$ and $(\mathbf{v}'^n)_{n \geq 1}$ are the sequences of vectors generated respectively by OFS and UFS, we have $\mathbf{v}^n = \text{sort}(\mathbf{v}'^n)$ for all n .

From now on, we consider a vector \mathbf{v} in $(\mathbb{R}^+)^d$ such that UFS never erases a coordinate, meaning that $\min_i v_i^n > 0$ for all n . We consider the infinite sequence $\Delta(\mathbf{v}) = (\delta_n)_{n \geq 1} \in \{1, \dots, d\}^\omega$ where δ_n is the index of the minimal coordinate of \mathbf{v}^n which is subtracted from all other ones. For all $n \geq 1$, we have $\mathbf{v}^n = \sigma_{\delta_{n-1}} \dots \sigma_{\delta_1}(\mathbf{v})$, and $\Omega^n = v_{\delta_1}^1 + \dots + v_{\delta_{n-1}}^{n-1}$. We set $\omega_n = v_{\delta_n}^n$ so that $\Omega^n = \sum_{i=1}^{n-1} \omega_i$. Note that, by hypothesis, $\omega_n > 0$ for all n . We have

$$\omega_n = v_{\delta_n}^n = \langle \mathbf{v}^n, \mathbf{e}_{\delta_n} \rangle = \langle \sigma_{\delta_{n-1}} \dots \sigma_{\delta_1}(\mathbf{v}), \mathbf{e}_{\delta_n} \rangle = \langle \mathbf{v}, {}^t\sigma_{\delta_1} \dots {}^t\sigma_{\delta_{n-1}}(\mathbf{e}_{\delta_n}) \rangle$$

where ${}^t\sigma$ denotes the transpose of σ . We set $\mathbf{T}_n = {}^t\sigma_{\delta_1} \dots {}^t\sigma_{\delta_{n-1}}(\mathbf{e}_{\delta_n})$ so that, for all n , we have $\langle \mathbf{v}, \mathbf{T}_n \rangle = \omega_n$. The sequence $(\mathbf{T}_n)_{n \geq 1}$ has some nice properties.

Lemma 7 ([7]). *If the first occurrence in $\Delta(\mathbf{v})$ of some $k \in \{1, \dots, d\}$ is at position n , meaning that $\delta_n = k$, and $\delta_i \neq k$ for all $i < n$, then $\mathbf{T}_1 + \dots + \mathbf{T}_n = \mathbf{e}_k$.*

Lemma 8 ([7]). *If m and n are the positions of two consecutive occurrences in $\Delta(\mathbf{v})$ of some $k \in \{1, \dots, d\}$, meaning that $m < n$, $\delta_m = \delta_n = k$ and $\delta_i \neq k$ for all $i \in \{m+1, \dots, n-1\}$, then $\mathbf{T}_{m+1} + \dots + \mathbf{T}_n = \mathbf{T}_m$.*

Lemma 9 ([7]). *If $i \leq j < k$ and $\delta_j \neq \delta_k$ then $\langle \mathbf{v}, (\mathbf{T}_1 + \dots + \mathbf{T}_k) + \mathbf{T}_i \rangle \geq \Omega^\infty$. Therefore, $(\mathbf{T}_1 + \dots + \mathbf{T}_k) + \mathbf{T}_i \notin \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$.*

We recall now the construction of the *geometric palindromic closure* of $\Delta(\mathbf{v})$ [7]. This construction builds incrementally a connected subset of \mathbb{Z}^d , which is easily shown to be included in $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$. We shall show that it is in fact exactly $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ when $\mathbf{v} \in \mathcal{K}_d$.

We define a sequence $(\mathbb{P}_n)_{n \geq 0}$ of subsets of \mathbb{Z}^d by:

$$\mathbb{P}_0 = \{\mathbf{0}\}, \text{ and } \mathbb{P}_n = \mathbb{P}_{n-1} \cup (\mathbb{P}_{n-1} + \mathbf{T}_n) \text{ for } n \geq 1.$$

Theorem 10 ([7]). $\mathbb{P}_\infty = \lim_{n \rightarrow \infty} \mathbb{P}_n$ is connected.

The set \mathbb{P}_∞ is the *geometric palindromic closure* of $\Delta(\mathbf{v})$ in \mathbb{Z}^d [7]. From the definition of \mathbb{P}_n , we get the following characterisation:

$$\mathbb{P}_n = \left\{ \sum_{i \in I} \mathbf{T}_i \mid I \subset \{1, \dots, n\} \right\} \quad \text{for all } n \geq 0;$$

$$\mathbb{P}_\infty = \left\{ \sum_{i \in I} \mathbf{T}_i \mid I \subset \mathbb{N}_1, |I| < \infty \right\}.$$

The inclusion $\mathbb{P}_\infty \subset \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is straightforward. From what precedes, each \mathbf{x} in \mathbb{P}_∞ may be written as $\mathbf{x} = \sum_{i \in I} \mathbf{T}_i$, for some finite subset I of \mathbb{N}_1 . Then

$\langle \mathbf{v}, \mathbf{x} \rangle = \sum_{i \in I} \langle \mathbf{v}, \mathbf{T}_i \rangle = \sum_{i \in I} \omega_i \in [0; \Omega(v)[$. Hence, \mathbf{x} belongs to $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$. In the sequel, we prove that we have also $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) \subset \mathbb{P}_\infty$, provided that each $k \in \{1, \dots, d\}$ occurs infinitely many times in $\Delta(\mathbf{v})$. The lemma below states that it is the case if $\mathbf{v} \in \mathcal{K}_d$.

Lemma 11. *If $\mathbf{v} \in \mathcal{K}_d$ then each $k \in \{1, \dots, d\}$ occurs infinitely many times in $\Delta(\mathbf{v})$.*

Proof. Assume, by contradiction, that $\mathbf{v} \in \mathcal{K}_d$, and some $k \in \{1, \dots, d\}$ does not occur anymore in $\Delta(\mathbf{v})$ for $n \geq n_0$. Since $\mathbf{v} \in \mathcal{K}_d \iff \mathbf{v}^{n_0} \in \mathcal{K}_d$, and $\Delta(\mathbf{v}) = \delta_1 \cdots \delta_{n_0-1} \Delta(\mathbf{v}^{n_0})$, we may assume, without loss of generality, that $n_0 = 1$, i.e. that k never occurs in $\Delta(\mathbf{v})$. Then for all $n > 1$, we have $v_k^n = v_k^{n-1} - \omega_n = v_k - \sum_{i=1}^{n-1} \omega_i = v_k - \Omega^n$. Since $\mathbf{v} \in \mathcal{K}_d$, we have $\lim_{n \rightarrow \infty} \mathbf{v}^n = \mathbf{0}$ so that $v_k = \lim_{n \rightarrow \infty} \Omega^n = \Omega^\infty = \|\mathbf{v}\|_1 / (d-1)$. But $v_k \leq \|\mathbf{v}\|_\infty$ and, by assumption, $\|\mathbf{v}\|_\infty < \|\mathbf{v}\|_1 / (d-1)$. Hence a contradiction. \square

From now on, we assume that \mathbf{v} belongs to \mathcal{K}_d so that each $k \in \{1, \dots, d\}$ occurs infinitely many times in $\Delta(\mathbf{v})$. To prove our main theorem, we still need some additional technical results. The first lemma below is an immediate consequence of the proof of Th. 13 in [7].

Lemma 12 ([7]). *For all \mathbf{x}, \mathbf{y} in \mathbb{P}_∞ , we have $\mathbf{x} - \mathbf{y} \in \mathbb{P}_\infty$ or $\mathbf{y} - \mathbf{x} \in \mathbb{P}_\infty$.*

Theorem 13. *Each \mathbf{x} in \mathbb{Z}^d may be written as $\pm(\alpha_1 \mathbf{T}_1 + \cdots + \alpha_m \mathbf{T}_m)$ for some $m \geq 0$ and $\alpha_1, \dots, \alpha_m \in \mathbb{N}_0$.*

Proof. Let $\mathbf{x} \in \mathbb{Z}^d$. We have $\mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_d \mathbf{e}_d$. From lemma 7, each \mathbf{e}_k may be written as $\sum_{j=1}^{n_k} \mathbf{T}_j$ where n_k is the index of the first occurrence of k in Δ . Then $\mathbf{x} = \sum_{j=1}^r y_j \mathbf{T}_j$ for some $r \geq 0$ and $y_1, \dots, y_r \in \mathbb{Z}$. Since \mathbf{T}_i 's belong to \mathbb{P}_∞ , this last sum may always be decomposed as $(\mathbf{U}_1 + \cdots + \mathbf{U}_p) - (\mathbf{V}_1 + \cdots + \mathbf{V}_q)$ for some $p, q \geq 0$ where \mathbf{U}_i 's and \mathbf{V}_i 's belong to \mathbb{P}_∞ . If $p > 0$ and $q > 0$ then either $\mathbf{U}_1 - \mathbf{V}_1 = 0$, in which case we may simply *remove* \mathbf{U}_1 and \mathbf{V}_1 from the sum, or by lemma 12, we have either $\mathbf{U}_1 - \mathbf{V}_1 = \mathbf{U}'_1$ where $\mathbf{U}'_1 \in \mathbb{P}_\infty$ or $\mathbf{U}_1 - \mathbf{V}_1 = -\mathbf{V}'_1$ where $\mathbf{V}'_1 \in \mathbb{P}_\infty$. Replacing $\mathbf{U}_1 - \mathbf{V}_1$ with either \mathbf{U}'_1 or $-\mathbf{V}'_1$, the number of terms in the sum decreases. Repeating this process as long as $p > 0$ and $q > 0$, yields an expression of \mathbf{x} as $\pm(\mathbf{W}_1 + \cdots + \mathbf{W}_s)$ for some $s \geq 0$ and $\mathbf{W}_1, \dots, \mathbf{W}_s \in \mathbb{P}_\infty$. Now, each \mathbf{W}_i may be written as a finite sum of \mathbf{T}_j 's. Doing so, and collecting the \mathbf{T}_j 's yields the result. \square

Corollary 14. *Each \mathbf{x} in $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ may be written as $\alpha_1 \mathbf{T}_1 + \cdots + \alpha_m \mathbf{T}_m$ for some $m \geq 0$ and $\alpha_1, \dots, \alpha_m \in \mathbb{N}_0$.*

Proof. We have $\mathbf{x} = \varepsilon \times (\alpha_1 \mathbf{T}_1 + \cdots + \alpha_m \mathbf{T}_m)$ for some $\varepsilon = \pm 1$, $m \geq 0$, and $\alpha_1, \dots, \alpha_m \in \mathbb{N}_0$. Then $\langle \mathbf{v}, \mathbf{x} \rangle = \varepsilon \times (\alpha_1 \langle \mathbf{v}, \mathbf{T}_1 \rangle + \cdots + \alpha_m \langle \mathbf{v}, \mathbf{T}_m \rangle) = \varepsilon \times (\alpha_1 \omega_1 + \cdots + \alpha_m \omega_m)$. Since $\mathbf{x} \in \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ we have $\langle \mathbf{v}, \mathbf{x} \rangle \geq 0$ so that $\varepsilon = +1$ because $\omega_i > 0$ for all i . \square

For any subset X of \mathbb{Z} containing 0, we denote by X^*0^ω the set of infinite sequences of which the terms belongs to X , and containing only finitely many non-zero terms. We define a mapping ψ from \mathbb{Z}^*0^ω to \mathbb{Z}^d by $\psi(W) = \sum_{i \geq 1} w_i T_i$. Then we have $\mathbb{P}_\infty = \psi(\{0, 1\}^*0^\omega) = \{\psi(W) \mid W \in \{0, 1\}^*0^\omega\}$. Next lemma is the immediate reformulation of Lemmas 7 and 8 in terms of ψ .

Lemma 15.

- If n is the index of the first occurrence of k in $\Delta(\mathbf{v})$, then $\psi(1^n 0^\omega) = e_k$.
- If m and n are the indexes of two consecutive occurrences of k in $\Delta(\mathbf{v})$, then $\psi(0^{m-1} 0 1^{n-m} 0^\omega) = \psi(0^{m-1} 1 0^{n-m} 0^\omega)$.

To prove that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) = \mathbb{P}_\infty$, it is sufficient to prove that each \mathbf{x} in $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ may be written as $\psi(W)$ for some W in $\{0, 1\}^*0^\omega$. Thanks to Cor. 14, we may find $W \in \mathbb{N}_0^*0^\omega$ such that $\mathbf{x} = \psi(W)$. Using Lemma 15, we shall transform W into $W' \in \{0, 1\}^*0^\omega$ such that $\psi(W') = \psi(W)$. We define two transformations on $\mathbb{N}_0^*0^\omega$ as follows.

$$\begin{array}{ccccccc} i & = & m & & n & & \\ \Delta & = & \cdots & k & \cdots \cdots \cdots & k & \cdots \end{array}$$

Reduction

$$\begin{array}{ccccccc} W & = & \cdots & 0 & w'_{m+1} + 1 & \cdots & w'_n + 1 & \cdots \\ \rightarrow W' & = & \cdots & 1 & w'_{m+1} & \cdots & w'_n & \cdots \end{array}$$

Expansion

$$\begin{array}{ccccccc} W & = & \cdots & u + 2 & w_{m+1} & \cdots & w_n & \cdots \\ \rightarrow W' & = & \cdots & u + 1 & w_{m+1} + 1 & \cdots & w_n + 1 & \cdots \end{array}$$

where $m < n$, and $u, w_{m+1}, \dots, w_n, w'_{m+1}, \dots, w'_n \geq 0$, and $\delta_i \neq k$ for all $i \in \{m + 1, \dots, n - 1\}$.

According to Lemma 15, both these transformations preserve $\psi(W)$. Given $W \in \mathbb{N}_0^*0^\omega$, we apply these two transformations with the following strategy.

First apply **Reduction** as much as possible.
 Then, as long as $w_i \geq 2$ for some i :

1. apply **Expansion** once at the last position m such that $w_m \geq 2$;
2. apply **Reduction** as much as possible.

This strategy, if it terminates, obviously yields a sequence in $\{0, 1\}^*0^\omega$ since otherwise **Expansion** would still apply.

Theorem 16. *If $W \in \mathbb{N}_0^*0^\omega$ and $\psi(W) \in \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, then applying **Reduction** and **Expansion** to W with the strategy above, terminates and yields $W' \in \{0, 1\}^*0^\omega$ such that $\psi(W') = \psi(W)$.*

Before proving this theorem, let us introduce some more notation. Given a sequence W in \mathbb{N}_0^* , $|W|$ is the length of W . If $W \in \mathbb{N}_0^*0^\omega$ then $|W|$ is the index of the last non-zero term in W , or 0 if $W = 0^\omega$.

Note to the reviewer:
[full proof](#) given
in the appendix.

Sketch of proof of Th. 16. We consider the multiset M of all terms in W which are greater than 2. Our strategy ensures that M never increases. It decreases each time an expansion is performed on $w_m \cdots w_n$ if $w_m \geq 3$. It also decreases when a reduction is performed on $w_m \cdots w_n$ if a position i exists between $m + 1$ and n such that $w_i \geq 3$. Hence, after finitely many transformation steps, for each transformation, we have $w_i \leq 2$ for all $i \in [m; n]$, and M does not evolve anymore. It is therefore sufficient to prove termination when $W \in \{0, 1, 2\}^* 0^\omega$.

We observe that each transformation decreases W in the lexicographic ordering induced by $1 < 2 < 0$, so that the transformation process never loops. Although this ordering is not a well-order on infinite sequences, it is on sequences with bounded length. Each transformation which does not increase $|W|$, may be seen as operating on $\{0, 1, 2\}^{|W|}$. Therefore, there may be only finitely many reductions between two expansions, and the transformation process may not terminate only if it performs infinitely many expansions which increase $|W|$.

Finally, let $\rho_2(W)$ be the number of maximal sub-sequences in W containing no 0, and containing at least one 2. A careful examination of the transformation rules shows that $\rho_2(W)$ never increases. A case analysis shows that each expansion step which increases $|W|$ is eventually followed by a reduction step which reduces $\rho_2(W)$. The transformation process therefore terminates. \square

From this theorem and Cor. 14, we deduce that each \mathbf{x} in $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ may be written as $\psi(W)$ with $W \in \{0, 1\}^* 0^\omega$, and therefore $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) \subset \mathbb{P}_\infty$.

Theorem 17. *Let $d \geq 2$. For all \mathbf{v} in \mathcal{K}_d , $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected.*

As a corollary we get the following result.

Corollary 18. *For all $\mathbf{v} \in \mathcal{K}_d$, we have $\dim_{\mathbb{Q}}(\mathbf{v}) = d$.*

Proof. Assume that $\dim_{\mathbb{Q}}(\mathbf{v}) < d$. Then there exists $\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{v}, \mathbf{p} \rangle = 0$. Thus, $\mathbf{p} \in \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, hence $\mathbf{p} = \psi(U)$ for some $U \in \{0, 1\}^* 0^\omega$. Since $\mathbf{p} \neq \mathbf{0}$, we have $U \neq 0^\omega$. Then $\langle \mathbf{v}, \mathbf{p} \rangle = \sum_{i \geq 1} u_i \langle \mathbf{v}, \mathbf{T}_i \rangle = \sum_{i \geq 1} u_i \omega_i > 0$. Hence a contradiction. \square

It should be noted that Kraaikamp & Meester actually proved Th. 3 with the assumption that $\dim_{\mathbb{Q}}(\mathbf{v}) = d$. However, they used this hypothesis only to ensure that $v_1^n > 0$ for all n . As a matter of fact, an earlier version of this theorem exists [10] with only the assumption that $v_1^n > 0$ for all n . With this weaker assumption, we get $\dim_{\mathbb{Q}}(\mathbf{v}) = d$ as a corollary.

5 Connectedness of Hyperplanes with non-zero shift

We have established that for each $d \geq 2$, $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected for all $\mathbf{v} \in \mathcal{K}_d$. The question which arises naturally is whether $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is still connected when $\mu \neq 0$. We already know that $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is connected for all $\mu \in \mathbb{R}$ if $\mathbf{v} \in \mathcal{K}_2$. For $d \geq 3$, we don't have a general result. However, it has been established in [2], for a specific vector in \mathcal{K}_3 , that $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is disconnected if $\mu = \Omega(\mathbf{v})$. Theorem 20 below shows that this holds for all $\mathbf{v} \in \mathcal{K}_d$, for all $d \geq 3$.

Lemma 19 ([7]). *Let $\Delta \in \{1, \dots, d\}^\omega$ and \mathbb{P}_∞ be the geometric palindromic closure of Δ . Then $\mathbb{P}_\infty \setminus \{0\}$ has exactly as many connected components as the cardinal of $\{\delta_i \mid i \in \mathbb{N}_1\}$.*

If $\mathbf{v} \in \mathcal{K}_d$ then each $k \in \{1, \dots, d\}$ occurs in $\Delta(\mathbf{v})$. Thus, $\mathbb{P}_\infty \setminus \{0\}$ has exactly d connected components, and is therefore disconnected since $d \geq 2$.

Theorem 20. *If $d \geq 3$ and $\mathbf{v} \in \mathcal{K}_d$ then $\mathbb{P}(\mathbf{v}, \Omega(\mathbf{v}), \Omega(\mathbf{v}))$ is disconnected.*

Proof. We have

$$\begin{aligned} & \mathbb{P}(\mathbf{v}, \Omega(\mathbf{v}), \Omega(\mathbf{v})) \\ &= \{\mathbf{x} \in \mathbb{Z}^d \mid 0 \leq \langle \mathbf{v}, \mathbf{x} \rangle + \Omega(\mathbf{v}) < \Omega(\mathbf{v})\} \\ &= \{\mathbf{x} \in \mathbb{Z}^d \mid -\Omega(\mathbf{v}) \leq \langle \mathbf{v}, \mathbf{x} \rangle < 0\} \\ &= -\{\mathbf{x} \in \mathbb{Z}^d \mid 0 < \langle \mathbf{v}, \mathbf{x} \rangle \leq \Omega(\mathbf{v})\} \\ &= -((\mathbb{P}(\mathbf{v}, \Omega(\mathbf{v}), \Omega(\mathbf{v})) \setminus \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{v}, \mathbf{x} \rangle = 0\}) \cup \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{v}, \mathbf{x} \rangle = \Omega(\mathbf{v})\}). \end{aligned}$$

Since $\mathbf{v} \in \mathcal{K}_d$, we have $\Omega(\mathbf{v}) = (v_1 + \dots + v_d)/(d-1)$, and by Cor. 18, $\dim_{\mathbb{Q}}(\mathbf{v}) = d$. Hence the only solution in \mathbb{Q} of $\langle \mathbf{v}, \mathbf{x} \rangle = 0$ is $\mathbf{v} = \mathbf{0}$, and the only solution of $\langle \mathbf{v}, \mathbf{x} \rangle = \Omega(\mathbf{v})$ is $x_1 = \dots = x_d = \frac{1}{d-1}$. Therefore, if $d \geq 3$, the equation $\langle \mathbf{v}, \mathbf{x} \rangle = \Omega(\mathbf{v})$ has no solution in \mathbb{Z}^d . Hence, $\mathbb{P}(\mathbf{v}, \Omega(\mathbf{v}), \Omega(\mathbf{v})) = -\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) \setminus \{\mathbf{0}\}$, which, by the Lemma 19, is disconnected. \square

6 Summary of results and perspectives

Let us now summarise the results of the previous sections about the connectedness of $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$.

Theorem 21. *Given $d \geq 2$ and $\mathbf{v} \in (\mathbb{R}^+)^d \setminus \{0\}$, we have the following results.*

- *If $\dim_{\mathbb{Q}}(\mathbf{v}) = 1$ then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is disconnected for all $\mu \in \mathbb{R}$.*
 - *If $\dim_{\mathbb{Q}}(\mathbf{v}) \geq 2$ then let $(\mathbf{v}^n)_{n \geq 1}$ be the sequence of vectors computed by the fully subtractive algorithm applied to \mathbf{v} . We have $\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{v})$ for all $\mu \in \mathbb{R}$, and $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected if and only if $\mathbf{v}^n \in \mathcal{K}_d$ for some n , and some $d' \leq d$. In this case:*
 - *If $d' = 2$ then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is connected for all $\mu \in \mathbb{R}$.*
 - *If $d' \geq 3$ then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is disconnected for some values of μ .*
- In particular, $\mathbb{P}(\mathbf{v}, \Omega(\mathbf{v}), \Omega(\mathbf{v}))$ is disconnected.*

This theorem provides a complete characterisation of the connectedness of $\mathbb{P}(\mathbf{v}, 0, \omega)$ at the critical thickness $\Omega(\mathbf{v}, 0)$. For the case $d = 3$, Th. 5.1 in [3] already established the connectedness of $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}, 0))$ for all $\mathbf{v} \in \mathcal{K}_3$. However, the *only if* part of that theorem is wrong. The authors claim falsely that the vectors in \mathcal{K}_3 are the only ones in $(\mathbb{R}^+)^3$ for which $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected. This is obviously false since $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected for all vectors of the form $(0, v_2, v_3)$ such that $\dim_{\mathbb{Q}}(v_2, v_3) = 2$. Their proof relies upon the fact that for $\mathbf{v} \in \mathcal{O}_3^+ \setminus \mathcal{K}_3$, we have eventually $v_3^n \geq v_1^n + v_2^n$ for some n . It indeed implies that $\mathbf{v}^n \notin \mathcal{K}_3$ but we might still have $\mathbf{v}^m \in \mathcal{K}_2$ for some $m > n$. It would imply that

$\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is disconnected only if $\dim_{\mathbb{Q}}(v_1^n, v_2^n) = 2$. Even if we exclude the trivial case above, we may simply take any vector \mathbf{v} such that $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 2$ and $v_1^n = 0$ for some n . Take for instance $\mathbf{v} = (1, 1, \sqrt{2}+1)$. We have $v_3 \geq v_1 + v_2$, but after one application of OFS, we get $\mathbf{v}^2 = (0, 1, \sqrt{2})$ which reduces to $\mathbf{v}^3 = (1, \sqrt{2})$. Now $\mathbb{P}(\mathbf{v}^3, 0, \Omega(\mathbf{v}^3))$ is connected so that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected.

In order to effectively determine whether $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected, in addition to Th. 21, we still need a way to decide whether the fully subtractive algorithm will eventually reach some \mathcal{K}_d . At present, we are not even able to decide whether a given vector $\mathbf{v} \in \mathbb{R}^d$ belongs to \mathcal{K}_d . This lets some open questions and some research directions. Given $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$ we are interested in the following questions:

- decide whether OFS will erase some coordinate, i.e. whether $v_1^n = 0$ for some n ;
- if not, decide whether \mathbf{v} belongs to \mathcal{K}_d ;
- if $\mathbf{v} \in \mathcal{K}_d$, characterise the values of μ for which $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is connected.

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A Extended proofs

Proof of Th. 5. For all $n \geq 1$ and all $r \in \{2, \dots, d\}$, we set

$$K_r^n = (r-1)v_r^n - (v_1^n + \dots + v_r^n).$$

In particular, we have $K_d^n = (d-1)\|\mathbf{v}^n\|_\infty - \|\mathbf{v}^n\|_1$. We observe that K_r^n is always increasing in r . Indeed, we have $K_{r+1}^n = K_r^n + (r-1)(v_{r+1}^n - v_r^n)$ for all $r \in 2, \dots, d-1$. Since \mathbf{v}^n is ordered in non decreasing order, we have $v_{r+1}^n - v_r^n \geq 0$, hence $K_{r+1}^n \geq K_r^n$.

As long as $v_1^n > 0$, K_r^n is also increasing in n for all r . Indeed, if $v_1^n > 0$, then $(v_1^{n+1}, \dots, v_d^{n+1}) = \text{sort}(v_1^n, v_2^n - v_1^n, \dots, v_d^n - v_1^n)$. We must distinguish two cases.

- If $r = d$ or $r < d$ and $v_1^n \leq v_{r+1}^n - v_1^n$, then $(v_1^{n+1}, \dots, v_r^{n+1}) = \text{sort}(v_1^n, v_2^n - v_1^n, \dots, v_r^n - v_1^n)$. In this case, we have

$$\begin{aligned} K_r^{n+1} - K_r^n &= (r-1)v_r^{n+1} - (v_1^{n+1} + \dots + v_r^{n+1}) - (r-1)v_r^n + (v_1^n + \dots + v_r^n) \\ &= (r-1)\max(v_r^n - v_1^n, v_1^n) - (v_1^n + \dots + v_r^n) + (r-1)v_1^n \\ &\quad - (r-1)v_r^n + (v_1^n + \dots + v_r^n) \\ &= (r-1)\max(0, 2v_1^n - v_r^n) \end{aligned}$$

This last quantity is obviously non negative so that $K_r^{n+1} \geq K_r^n$.

- If $r < d$ and $v_1^n > v_{r+1}^n - v_1^n$, then $(v_1^{n+1}, \dots, v_r^{n+1}) = (v_2^n - v_1^n, \dots, v_{r+1}^n - v_1^n)$. In this case, we have

$$\begin{aligned} K_r^{n+1} - K_r^n &= (r-1)v_r^{n+1} - (v_1^{n+1} + \dots + v_r^{n+1}) - (r-1)v_r^n + (v_1^n + \dots + v_r^n) \\ &= (r-1)(v_{r+1}^n - v_1^n) - (v_2^n + \dots + v_{r+1}^n) + r v_1^n \\ &\quad - (r-1)v_r^n + (v_1^n + \dots + v_r^n) \\ &= (r-1)(v_{r+1}^n - v_r^n) + (2v_1^n - v_{r+1}^n) \end{aligned}$$

Since \mathbf{v}^n is ordered, and $2v_1^n - v_{r+1}^n > 0$ by assumption, this last quantity is positive. Therefore, $K_r^{n+1} > K_r^n$.

Now, let r_0 be the smallest index r such that $K_r^n \geq 0$ for some n . Such an r exists since $K_d^n \geq 0$ for some n by assumption. Since $K_2^n = -v_1^n < 0$, we have $r_0 \geq 3$. Let n_0 be the smallest index n such that $K_{r_0}^n \geq 0$. Then we have $\dim_{\mathbb{Q}}(v_1^{n_0}, \dots, v_{r_0-1}^{n_0}) \geq 2$. Indeed, for all $n \geq n_0$, we have $v_1^n < v_{r_0}^n - v_1^n$. Otherwise, since $\mathbf{v}^{n+1} = \text{sort}(v_1^n, v_2^n - v_1^n, \dots, v_d^n - v_1^n)$, we would have

$$(v_1^{n+1}, \dots, v_{r_0-1}^{n+1}) = (v_2^n - v_1^n, \dots, v_{r_0-1}^n - v_1^n, v_{r_0}^n - v_1^n),$$

hence,

$$K_{r_0-1}^{n+1} = K_{r_0}^n + (2v_1^n - v_{r_0}^n) \geq K_{r_0}^n \geq 0.$$

This would contradict the minimality of r_0 . Therefore,

$$(v_1^{n+1}, \dots, v_{r_0-1}^{n+1}) = \text{sort}(v_1^n, v_2^n - v_1^n, \dots, v_{r_0-1}^n - v_1^n),$$

and $((v_1^{n_0+k-1}, \dots, v_{r_0-1}^{n_0+k-1}))_{k \geq 1}$ is the sequence of vectors obtained when applying OFS to $(v_1^{n_0}, \dots, v_{r_0-1}^{n_0})$. If $\dim_{\mathbb{Q}}(v_1^{n_0}, \dots, v_{r_0-1}^{n_0}) = 1$ then eventually, we have $v_1^n = 0$ for some n , which contradicts the hypothesis.

In addition, since $(r_0 - 1)v_{r_0-1}^n - (v_1^n, \dots, v_{r_0-1}^n) < 0$ for all n , we have

$$\Omega(v_1^{n_0}, \dots, v_{r_0-1}^{n_0}) = \frac{v_1^{n_0} + \dots + v_{r_0-1}^{n_0}}{r_0 - 2} \leq v_{r_0}^{n_0}.$$

By Lemma. 6, we get $\Omega(\mathbf{v}^{n_0}) = v_d^{n_0} = \|\mathbf{v}^{n_0}\|_{\infty}$. Then, by Th. 1, $\mathbb{P}(\mathbf{v}^{n_0}, \mu, \Omega(\mathbf{v}^{n_0}))$ is disconnected for all $\mu \in \mathbb{R}$. Therefore, $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is also disconnected for all $\mu \in \mathbb{R}$. \square

Proof of Th. 16. Let us start with a remark which will be used freely in the rest of this proof. When some δ is repeated in Δ , which means $\Delta = \Delta_0 \delta^p \Delta'$ for some $p \geq 2$, Lemma 15 applies at each position m between $|\Delta_0| + 1$ and $|\Delta_0| + p - 1$ with $n = m + 1$. This means that for any U, V, V' in \mathbb{N}_0^* such that $|U| = |\Delta_0|$, $|V| = |V'| = p$, and $W \in \mathbb{N}_0^* 0^\omega$, we have $\psi(U V W) = \psi(U V' W)$ if and only if $\sum_{i=1}^p v_i = \sum_{i=1}^p v'_i$. In effect, when applied at these positions, **Reduction** and **Expansion** simply *move* one unit by one position backward or forward.

We consider the multiset M of all terms in W which are greater than 2. Our strategy ensures that M never increases. It decreases each time an expansion is performed on $w_m \cdots w_n$ if $w_m \geq 3$. It also decreases when a reduction is performed on $w_m \cdots w_n$ if a position i exists between $m + 1$ and n such that $w_i \geq 3$. Hence, after finitely many transformation steps, for each transformation, we have $w_i \leq 2$ for all $i \in [m; n]$, and M does not evolve anymore. It is therefore sufficient to prove termination when $W \in \{0, 1, 2\}^* 0^\omega$.

We observe that each transformation decreases W in the lexicographic ordering induced by $1 < 2 < 0$, so that the transformation process never loops. Although this ordering is not a well-order on infinite sequences, it is on sequences with bounded length. If $W \in \{0, 1, 2\}^r 0^\omega$ for some r , and W' is obtained from W by one reduction step, then we also have $W' \in \{0, 1, 2\}^r 0^\omega$. Thus we may see **Reduction** as operating on $\{0, 1, 2\}^r$. We may therefore have only finitely many reductions between two expansions. Also, if W' is obtained from W by one expansion step, and $|W'| = |W|$ then $W' < W$ in the lexicographic ordering on $\{0, 1, 2\}^{|W|}$. Hence, the only way for the transformation process not to terminate is to perform infinitely many expansions which increase $|W|$.

Let us first consider the particular case where m is the position of the last 2 in W , and $w_i \geq 1$ for all $i < m$. Given $U \in \mathbb{N}_0^*$, we denote by $|U|_2$ the number of 2's in U . Since $\psi(W) \in \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, and no reduction applies, by Lemma 9, Δ and W must have the following form:

$$\begin{aligned} \Delta &= \Delta_0 \delta^{1+n_0+n'_0+n''_0} \delta' \Delta' \\ W &= 1^{|\Delta_0|} U \mathbf{2} 1^{n'_0} 0^{n''_0} w V \end{aligned}$$

where $n_0, n'_0, n''_0 \geq 0$, $\delta' \neq \delta$, $w \in \{0, 1\}$, $V \in \{0, 1\}^* 0^\omega$, $U \in \{1, 2\}^{n_0}$, and $|U|_2 \leq n''_0$. Otherwise, we would have $\langle \mathbf{v}, \psi(W) \rangle \geq \Omega(\mathbf{v})$, hence $\psi(W) \notin \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$.

If $|U|_2 < n''_0$, then, according to the remark at the beginning of this proof, after finitely many expansions, we get $W' = 1^{|\Delta_0|} 1^{n_0+n'_0+|U|_2+1} 0^{n''_0-|U|_2-1} w V$, and we are done. If $|U|_2 = n''_0$ then we get $W' = 1^{|\Delta_0|} 1^{n_0+n'_0+n''_0} 2 w V$. In this case, still by Lemma 9, we must have

$$\begin{aligned} \Delta &= \Delta_0 \delta^{m_0+1} \prod_{i=1}^r (\Delta_i \delta^{n_i+1}) \Delta_{r+1} \delta^{n_{r+1}+1} \Delta'' \\ W' &= 1^{|\Delta_0|} 1^{m_0} 2 \prod_{i=1}^r (0^{|\Delta_i|} 1^{n_i+1}) 0^{|\Delta_{r+1}|} 1^p 0^{n_{r+1}-p} V' \end{aligned}$$

where $r \geq 0$, δ does not occur in $\Delta_1, \dots, \Delta_{r+1}$, $m_0, n_1, \dots, n_{r+1} \geq 0$ and $0 \leq p \leq n_{r+1}$. Again, after finitely many expansions, we get

$$W'' = 1^{|\Delta_0|} 1^{m_0} 1 \prod_{i=1}^r (1^{|\Delta_i|} 1^{n_i+1}) 1^{|\Delta_{r+1}|} 1^p 1 0^{n_{r+1}-p} V'$$

and we are done since there is no 2 left in W'' .

We are now left with the case where, at each expansion step, a 0 occurs at some position before the last 2 in W . Let $\rho_2(W)$ be the number of maximal sub-sequences in W containing no 0, and containing at least one 2. A careful examination of the transformation rules shows that $\rho_2(W)$ never increases. We shall show that $\rho_2(W)$ eventually decreases after each expansion which increases $|W|$, and the process must therefore terminate.

Consider the n -th expansion step, and assume that $|W|$ increases. Let i_n, j_n, i'_n and j'_n be respectively: the position of the last 2 in W , the position of the last 0 before this 2, the position of the next occurrence of δ_{i_n} in Δ , and the position of the next occurrence of δ_{j_n} in Δ . We have $\delta_{j_n} \neq \delta_{i_n}$ and $j'_n > i_n$, otherwise a reduction would be possible at position j_n . We set $\ell_0 = |W|$. Since we assumed that $|W|$ increases, we have $\ell_0 < i'_n$. The situation is as follows:

$$\begin{array}{ccccccc} & & & & \longleftarrow j'_n \longrightarrow & & \\ & & & & \ell_0 & & i'_n \\ \Delta & = & \Delta_1 \delta_{j_n} & \Delta_2 \delta_{i_n} & \Delta_3 & \Delta_4 \delta_{i_n} & \Delta' \\ W & = & W_1 0 & W_2 + 1 \ 2 & W_3 & 0^{n_4} 0 & 0^\omega \\ \mathbf{Exp.} \rightarrow W' & = & W_1 0 & W_2 + 1 \ 1 & W_3 + 1 & 1^{n_4} 1 & 0^\omega \end{array}$$

where $W_1 \in \{0, 1, 2\}^*$, $W_2, W_3 \in \{0, 1\}^*$, δ_{j_n} does not occur in Δ_2 , and δ_{i_n} does not occur in $\Delta_3 \Delta_4$. By a slight abuse of notation, if $U \in \mathbb{N}_0^*$, we write $U + 1$ for $U + 1^{|U|}$, the sequence obtained by adding 1 to each term of U .

The idea is to perform expansions until either no 2 is left in W after position j_n , in which case $\rho_2(W)$ has decreased and we are done, or we eventually reach j'_n , in which case a reduction is possible at position j_n . Notice that no reduction may occur until then because it would have been possible before the expansion.

Let us consider the *base case* where $W_3 \in 1^*$, which means that no 0 occurs in W between the position of the last 2 and ℓ_0 . We must have $j'_n > \ell_0$. Otherwise, a reduction would be possible at position j_n . Let $W_3 = 1^{n_3}$ for some $n_3 \geq 0$. We distinguish two cases according to the position of j'_n with respect to i'_n .

(i) $\ell_0 < j'_n < i'_n$.

$$\begin{array}{l}
\Delta = \Delta_1 \delta_{j_n} \quad \Delta_2 \quad \delta_{i_n} \quad \Delta_3 \quad \ell_0 \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right. \\
W = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{2} \quad 1^{n_3} \quad \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right. \\
\mathbf{Exp.} \rightarrow W' = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{1} \quad \mathbf{2}^{n_3} \quad \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right. \\
\mathbf{Red.} \rightarrow W'' = W_1 \quad \mathbf{1} \quad W_2 \quad \mathbf{0} \quad 1^{n_3} \quad \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right.
\end{array}$$

In this case, a reduction follows immediately the expansion, and *reduces* all 2's after position j_n . Then $\rho_2(W)$ decreases by one.

(ii) $i'_n < j'_n$.

$$\begin{array}{l}
\Delta = \Delta_1 \delta_{j_n} \quad \Delta_2 \quad \delta_{i_n} \quad \Delta_3 \quad \ell_0 \left| \begin{array}{c} i'_n \quad j'_n \\ \Delta_4 \quad \delta_{i_n} \quad \Delta_5 \quad \delta_{j_n} \quad \Delta' \end{array} \right. \\
W = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{2} \quad 1^{n_3} \quad \left| \begin{array}{c} i'_n \quad j'_n \\ \Delta_4 \quad \delta_{i_n} \quad \Delta_5 \quad \delta_{j_n} \quad \Delta' \end{array} \right. \\
\mathbf{Exp.} \rightarrow W' = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{1} \quad \mathbf{2}^{n_3} \quad \left| \begin{array}{c} i'_n \quad j'_n \\ \Delta_4 \quad \delta_{i_n} \quad \Delta_5 \quad \delta_{j_n} \quad \Delta' \end{array} \right.
\end{array}$$

In this case, expansion has not yet reached j'_n so that no reduction applies immediately. If $W_2 = 0^{n_2}$ for some $n_2 \geq 0$ and $n_3 = 0$, there is no 2 left after position j_n , and $\rho_2(W)$ has decreased by one. Otherwise, W' still has the same form. A new expansion is possible with a new i_{n+1} and a new i'_{n+1} but j and j' do not change, which means $j_{n+1} = j_n$ and $j'_{n+1} = j'_n$. Expansion continues without reduction until either we have no 2 left after j_n , in which case $\rho_2(W)$ has decreased by one, or we reach j'_n . Then we are in case (i) above.

For the general case, we have three possibilities, depending on the position of j'_n with respect to ℓ_0 and i'_n .

(i') $i'_n < j'_n$.

$$\begin{array}{l}
\Delta = \Delta_1 \delta_{j_n} \quad \Delta_2 \quad \delta_{i_n} \quad \Delta_3 \quad \ell_0 \left| \begin{array}{c} i'_n \quad j'_n \\ \Delta_4 \quad \delta_{i_n} \quad \Delta_5 \quad \delta_{j_n} \quad \Delta' \end{array} \right. \\
W = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{2} \quad W_3 \quad \left| \begin{array}{c} i'_n \quad j'_n \\ \Delta_4 \quad \delta_{i_n} \quad \Delta_5 \quad \delta_{j_n} \quad \Delta' \end{array} \right. \\
\mathbf{Exp.} \rightarrow W' = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{1} \quad \mathbf{W_3 + 1} \quad \left| \begin{array}{c} i'_n \quad j'_n \\ \Delta_4 \quad \delta_{i_n} \quad \Delta_5 \quad \delta_{j_n} \quad \Delta' \end{array} \right.
\end{array}$$

After the expansion, if W_3 is empty then $\rho_2(W)$ has decreased. Otherwise, we are in the base case, and $\rho_2(W)$ will eventually decrease.

(ii') $\ell_0 < j'_n < i'_n$.

$$\begin{array}{l}
\Delta = \Delta_1 \delta_{j_n} \quad \Delta_2 \quad \delta_{i_n} \quad \Delta_3 \quad \ell_0 \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right. \\
W = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{2} \quad W_3 \quad \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right. \\
\mathbf{Exp.} \rightarrow W' = W_1 \quad 0 \quad W_2 + 1 \quad \mathbf{1} \quad \mathbf{W_3 + 1} \quad \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right. \\
\mathbf{Red.} \rightarrow W'' = W_1 \quad \mathbf{1} \quad W_2 \quad \mathbf{0} \quad W_3 \quad \left| \begin{array}{c} j'_n \quad i'_n \\ \Delta'_4 \quad \delta_{j_n} \quad \Delta''_4 \quad \delta_{i_n} \quad \Delta' \end{array} \right.
\end{array}$$

In this case, a reduction follows immediately the expansion, and *reduces* all 2's after position j_n . Then $\rho_2(W)$ decreases by one.

(iii') $i_n < j'_n \leq \ell_0$.

$$\begin{array}{l}
 \Delta = \Delta_1 \delta_{j_n} \quad \Delta_2 \quad \delta_{i_n} \quad \Delta'_3 \quad \delta_{j'_n} \quad \Delta''_3 \quad \ell_0 \mid \Delta_4 \delta_{i'_n} \Delta' \\
 W = W_1 \quad 0 \quad W_2 + 1 \quad 2 \quad W'_3 \quad w_{j'_n} \quad W''_3 \quad 0^{n_4} \quad 0 \quad 0^\omega \\
 \mathbf{Exp.} \rightarrow W' = W_1 \quad 0 \quad W_2 + 1 \quad 1 \quad W'_3 + 1 \quad w_{j'_n} + 1 \quad W''_3 + 1 \quad 1^{n_4} \quad 1 \quad 0^\omega \\
 \mathbf{Red.} \rightarrow W'' = W_1 \quad 1 \quad W_2 \quad 0 \quad W'_3 \quad w_{j'_n} \quad W''_3 + 1 \quad 1^{n_4} \quad 1 \quad 0^\omega
 \end{array}$$

If W''_3 is empty then all 2's after position j_n have been *reduced*, and $\rho_2(W)$ has decreased by one. Otherwise, we are again in the base case.

Therefore, in all cases, after an expansion which increases $|W|$, $\rho_2(W)$ eventually decreases. This proves that the transformation process terminates. \square
