Patterns in Smooth Tilings*

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– Dedicated to the memory of Alberto del Lungo

Abstract

Smooth words are infinite words connected with the one defined by Kolakoski. This class is obtained by a bijective map on the free monoid over $\Sigma = \{1, 2\}$ that shows some surprising mixing properties and also permits the construction of tilings composed of infinite smooth words. We study the problem of the existence of some patterns in these tilings.

1 Introduction

Following the talk given in the last year's *Journées Montoises* [2], we analyze the existence of some patterns in the class of what we called *smooth tilings*, tilings made of two basic square tiles. This class of tilings is constructed from infinite words belonging to a class which is invariant under the action of the run-length encoding operator Δ, and is related to the Kolakoski word

$$K = 2211212212211211221211212211211212212211212212112112 2\cdots$$

After the pioneering work of Dekking [8, 9] who stated some mind-blowing conjectures that still remain unsolved, some efforts were devoted to the study of patterns in K. For instance we know from Arthuro's paper [6] that K does contain only a finite number of squares, implying by an inspection that K is cube-free, a result that was extended by the second author [5] to an infinite class $\mathcal{K}$ of words over $\Sigma^* = \{1, 2\}$ sharing the (conjectured) same factors with K, and for which the Dekking's conjectures can be restated.

The class $\mathcal{K}$ with the operator Δ can be viewed as a dynamical system topologically conjugate to the full shift (Σ^ω, σ), where σ is the shift operator (see [14, 10]). Indeed, there is a bijection $\Phi : \mathcal{K} \longrightarrow \Sigma^\omega$ such that each tiling is completely determined by a word in Σ^ω. This point of view gives rise in a natural way to the smooth tilings, allowing a new approach in order to understand the combinatorial structure of words in $\mathcal{K}$.

In this paper we study the two-dimensional patterns appearing in the tilings. We give first an algorithm that computes the minimum distance of a pattern from the origin. Then we look at the special case of periodic tilings and finally to the tilings corresponding to the fixed points of the map Φ.

2 Definitions and notations

Let us consider a finite *alphabet* of *letters* Σ. A *word* is a finite sequence of letters $w : [1..n] \longrightarrow \Sigma$, $n \in \mathbb{N}$, of length n, and $w[i]$ or w_i will denote its i-th letter, depending on the context in order to avoid confusion. The set of n-length words over Σ is denoted Σ^n. By convention the *empty* word is denoted ϵ and its length

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is 0. The free monoid generated by Σ is defined by $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$. The set of right infinite words is denoted by Σ^ω and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. Given a word $w \in \Sigma^*$, a *factor* f of w is a word $f \in \Sigma^*$ satisfying

$$\exists\, x, y \in \Sigma^*, w = xfy.$$

If $x = \epsilon$ (resp. $y = \epsilon$) then f is called *prefix* (resp. *suffix*). The set of all factors of w is denoted by $F(w)$, and those of length n is $F_n(w) = F(w) \cap \Sigma^n$. Finally $\mathrm{Pref}(w)$ denotes the set of all prefixes of w. The length of a word w is $|w|$, and the number of occurrences of a factor $f \in \Sigma^*$ is $|w|_f$. Clearly, the length of a word is given by the number of its letters, $|w| = \sum_{\alpha\in\Sigma} |w|_\alpha$.

The *mirror image* $\widetilde{u}$ of $u = u_1u_2\cdots u_n$ is the word $\widetilde{u} = u_n \cdots u_2u_1$. A *palindrome* is a word p such that $p = \widetilde{p}$. A factor of the form uu is called a square. For a language $L \subseteq \Sigma^\infty$, we denote by $\mathrm{Pal}(L)$ and $\mathrm{Squares}(L)$, the sets, respectively, of its palindromes and square finite factors. Over the restricted alphabet $\Sigma = \{1, 2\}$, there is a usual length preserving morphism, the swapping of the letters, defined by $\overline{1} = 2$; $\overline{2} = 1$, which extends to words as follows. The *complement* of $u = u_1u_2\cdots u_n \in \Sigma^n$, is the word $\overline{u} = \overline{u_1}\ \overline{u_2}\ \overline{u_3} \cdots \overline{u_n}$.

The occurrences of factors play an important role and an infinite word w is recurrent if it satisfies the condition

$$u \in F(w) \implies |w|_u = \infty\ .$$

Clearly, every periodic word is recurrent, and there exist recurrent but non-periodic words, such as the Thue-Morse word and the Sturmian words.

3 Run-length encoding

From here, we restrict our study to words over the alphabet $\Sigma = \{1, 2\}$. Every word $w \in \Sigma^*$ can be uniquely written as a product of factors as follows

$$w = \alpha^{i_1}\overline{\alpha}^{i_2}\alpha^{i_3}\overline{\alpha}^{i_4}\cdots = \prod_{k\in I} \alpha^{i_k}\overline{\alpha}^{i_{k+1}}, \quad \text{for } \alpha \in \Sigma,\ I = \{1, 2, \ldots, \lfloor |w|/2 \rfloor\},$$

where all exponents are positive except possibly the last one. We now define the run-length encoding operator $\Delta : \Sigma^* \longrightarrow \mathbb{N}^*$ by

$$\Delta(w) = i_1i_2i_3i_4\cdots = \prod_{k\geq 1} i_k.$$

This operator admits an extension, also denoted by Δ, to infinite words $\Delta : \Sigma^\omega \longrightarrow \mathbb{N}^\omega$.

Example. Let $\Sigma = \{1, 2\}$, and $w = 1221121221$. Then $w = 1^{\mathbf{1}}2^{\mathbf{2}}1^{\mathbf{2}}2^{\mathbf{1}}1^{\mathbf{1}}2^{\mathbf{2}}1^{\mathbf{1}}$, and therefore

$$\Delta(w) = \mathbf{1221121}.$$

Since the function Δ is not bijective ($\Delta(w) = \Delta(\overline{w})$), pseudo-inverse functions

$$\Delta_1^{-1}, \Delta_2^{-1} : \Sigma^* \longrightarrow \Sigma^*$$

are defined by

$$\Delta_\alpha^{-1}(u) = \alpha^{u[1]}\overline{\alpha}^{u[2]}\alpha^{u[3]}\overline{\alpha}^{u[4]}\cdots, \quad \text{for } \alpha \in \{1, 2\}.$$

Example. Let $w = 1221121$. Then

$$\Delta_1^{-1}(w) = 1221121221, \quad \text{and} \quad \Delta_2^{-1}(w) = 2112212112.$$

As shown for instance in [4], the operator Δ can be iterated and the set of *smooth* words is

$$\mathcal{K} = \{w \in \Sigma^\omega \mid \forall k \in \mathbb{N}, \Delta^k(w) \in \Sigma^\omega\}.$$

A bijection $\Phi : \mathcal{K} \longrightarrow \Sigma^\omega$ is then defined by

$$\Phi(w)[j+1] = \Delta^j(w)[1] \quad \text{for} \quad 0 \leq j \leq k,$$

and its inverse is inductively defined as follows. Let $u \in \Sigma^+$, then

$$\Phi^{-1}(u) = w_1 \ ,$$

where

$$w_n = \begin{cases} u[|u|], & \text{if } n = |u|; \\ \Delta^{-1}_{u[n]}(w_{n+1}) & \text{if } 1 \leq n < |u|. \end{cases}$$

Example. Let $w = 1221121221$. The successive application of Δ gives:

$$\begin{aligned}
\Delta^0(w) &= \mathbf{1}221121221 \\
\Delta^1(w) &= \mathbf{1}221121 \\
\Delta^2(w) &= \mathbf{1}2211 \\
\Delta^3(w) &= \mathbf{1}22 \\
\Delta^4(w) &= \mathbf{1}2 \\
\Delta^5(w) &= \mathbf{1}1 \\
\Delta^6(w) &= \mathbf{2}
\end{aligned}$$

Hence, $\Phi(w) = \Delta^0(w)[1]\,\Delta^1(w)[1]\,\Delta^2(w)[1]\,\Delta^3(w)[1]\,\Delta^4(w)[1]\,\Delta^5(w)[1]\,\Delta^6(w)[1] = \mathbf{1111112}$. We can also find inductively, starting from the bottom, that $\Phi^{-1}(\Phi(w)) = 1221121221$.

4 Smooth tilings

As shown in [2, 3], the plane can be filled with 1's and 2's using the inverse of the bijection Φ. More precisely, a tiling of the discrete plane is a function

$$T : \mathbb{Z} \times \mathbb{Z} \times \Sigma^* \longrightarrow \Sigma$$

defined as follows. The discrete plane is splitted in 2 halfplanes

$$(\mathbb{Z}^+ \times \mathbb{Z}) \bigcup (\mathbb{Z}^- \times \mathbb{Z}),$$

and the tiling is constructed as follows. Let $u \in \Sigma^*$, and according to the definition of the inverse of Φ, let

$$w_n = \begin{cases} u[|u|], & \text{if } n = |u|; \\ \Delta^{-1}_{u[n]}(w_{n+1}) & \text{if } 1 \leq n < |u|. \end{cases}$$

Define the matrix composed of the w_n by

$$\forall i, 1 \leq i \leq |u|, \quad \text{and} \quad \forall j, 1 \leq j \leq |w_i|, \quad M_u[i,j] = w_i[j] \ .$$

Finally the tiling of the plane is obtained by constructing the matrix $M_{\overline{u}}$ and glueing it to form the (matrix of the) tiling

$$T_u[i,j] = \begin{cases} M_u[i,j] & \text{if } j > 0; \\ M_{\overline{u}}[i, 1-j] & \text{if } j \leq 0; \end{cases}$$

Clearly this tiling extends to bi-infinite words $u \in {}^\omega\Sigma^\omega$, where M_u and $M_{\overline{u}}$ correspond respectively to the positive and negative half planes.

Example. Let $w = (1121)^\omega = 11211121112111211121\cdots$ and $p = (1121)^4$ a prefix of w. Consider the bi-infinite word $u = \widetilde{w}\,w$. The tiling is obtained by writing vertically $\widetilde{p}\,p$ of w. First the righthand side is filled from the bottom. Then we write $\overline{\widetilde{p}\,p}$ to the left of p and fill the lefthand side from the bottom.

2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2
1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2
1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2
2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2
1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2
1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2
2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2
1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2
1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2
1	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1	2	1	2	2	1
2	2	1	2	2	1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	2	2
2	1	2	2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	1	1	2	1	1	2	1	2	2	1	2	2	1
1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	**1**	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2
2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	**1**	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2
1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	**2**	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2
1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	**1**	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	**1**	2	1	1	2	1	1	2	2	1	2	1	1	2	1	2
2	2	1	2	2	1	2	1	1	2	1	1	2	2	1	2	**1**	1	2	1	2	2	1	1	2	1	1	2	2	1	2	2
1	2	1	1	2	2	1	2	1	1	2	1	2	2	1	1	**2**	1	1	2	2	1	2	2	1	2	1	1	2	1	1	2
1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	**1**	2	2	1	2	1	1	2	1	2	2	1	1	2	1	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	**1**	2	1	1	2	1	1	2	2	1	2					
					1	2	1	1	2	1	1	2	2	1	2	**1**	1	2	1	2	2	1	1								
								1	1	2	1	2	2	1	1	**2**	1	1	2	2											
											2	1	1	2	2	**1**	2	2													
													1	2	2	**1**	2														
														1	2	**1**	1														
														1	1	**2**															
															2	**1**															

Tilings are then obtained by replacing 2 with ■ , and leaving 1 blank.

5 Patterns in smooth tilings

We consider first small patterns, and then, some properties of periodic as well as fix-point tilings are described. By *smooth pattern* we mean a subset, finite or not, of some smooth tiling.

5.1 Small patterns in smooth tilings

In order to estimate the size of the search of given patterns in infinite smooth tilings we start with an exact formula for the number of cube free words of length n. This gives an upper bound for the number of smooth factors of length n. This is then applied to the $n \times n$ square patterns. Then, we make a systematic search of small $n \times n$ smooth patterns appearing in some tilings. Finally, two-dimensional smooth palindromic patterns are analysed and a conjecture is made concerning the non existence of such patterns in arbitrary smooth tilings.

Bounds for the number of smooth $n \times n$ square patterns

Let $\mathcal{F} = \{111, 222\}$ be a set of forbidden patterns and let $\mathcal{C}$ be the set of infinite words avoiding patterns in $\mathcal{F}$. Since every factor of a smooth word is cube free [5], we have the trivial bounds

$$|F_n(\mathcal{K})| \leq |F_n(\mathcal{C})|$$

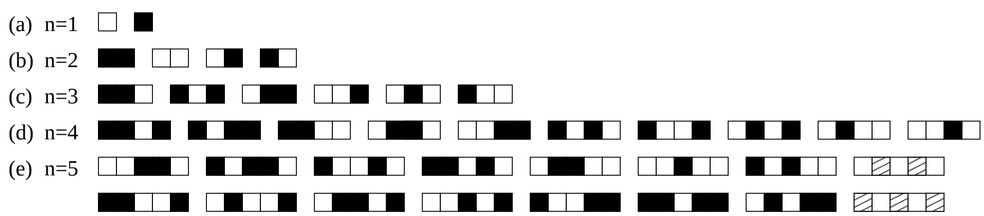
where $\mathcal{K}$ denotes the set of smooth words. We use the notation $f_n \leq c_n$, for sake of simplicity. Since the number s_n of smooth square patterns is f_n^n, we have the obvious estimate

$$s_n \leq c_n^n.$$

The generating series of c_n is

$$C(x) = \sum_{n\geq 0} c_n x^n = \frac{2}{1-x-x^2} - 1 = \sum_{n\geq 0} 2\cdot \mathrm{Fib}(n) x^n - 1 = 1 + 2x + 4x^2 + 6x^3 + 10x^4 + 16x^5 \ldots.$$

Remark that, for $n = 1 \ldots 4$, the number c_n of cube free words is equal to the number f_n of smooth factors, but this is no longer true for $n \geq 5$:



Proposition 1 *The number s_n of smooth $n \times n$ square patterns satisfies*

$$\begin{aligned} s_n \leq (2\cdot \mathrm{Fib}(n))^n &= \left(\frac{2}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)^n \\ &\sim \left(\frac{2}{\sqrt{5}}\right)^n \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

We also have the asymptotic value $\frac{s_n}{c_n^n} \to 0$, that is, the fraction of smooth $n \times n$ square patterns among the cube free $n \times n$ square patterns tends to 0 as n goes to infinity.

Occurrences

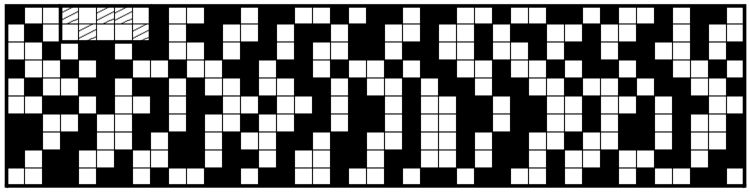
A problem of interest is to know whether a pattern P occurs in a smooth tiling T or not. This introduces the notion of *index of first occurence*, denoted $I_{\min}(P, u)$, between the vertical axis (i.e the vertical word u) and the first occurrence of the pattern if it exists. In other words, the problem can be defined as follows: *For a given pattern composed of n superimposed smooth factors $u_1, u_2, \ldots, u_n$, determine a tiling (i.e. a word u such that the distance from u is minimal.*

Example. Consider two smooth factors, $u_1 = \mathbf{21221}$ and $u_2 = \mathbf{12112}$. Since Δ is a contracting map, we must consider 2 cases:

(a) Take first, if it exists, the greatest suffix of u_1 which is coded by a prefix of u_2 as shown below

2	1	1	**2**	**1**	**2**	**2**	**1**	2	1	1	2	2	1	2
1	2	1	**1**	**2**	**1**	**1**	**2**	2	1					
1	1	2	1	2	2	1								
2	1	1	2											
1	2													
1														

Then, try extend to the left and down, keeping smoothness, until a vertical word is obtained. If this is possible a complete infinite smooth tiling can be constructed :

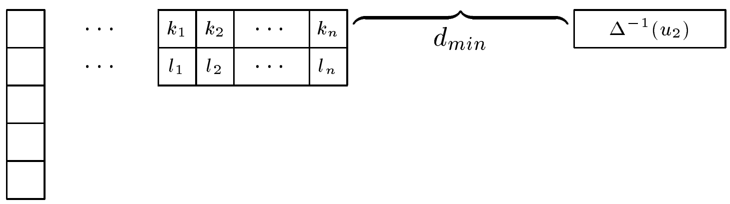


If such a suffix does not exist, then we consider the second case:

(b) In this case, the configuration

$$\begin{aligned} u_1 &= k_1 k_2 \dots k_n \\ u_2 &= l_1 l_2 \dots l_n, \qquad k_i, l_i \in \{1, 2\} \end{aligned}$$

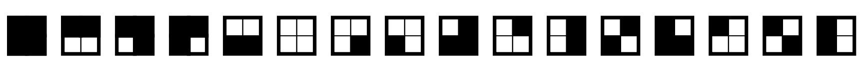
does not contain a suffix s of u_1 and a prefix p of u_2, such that $\Delta(s) = p$. Therefore, the only possible configuration is shown below,



where d_{min} represents the minimal distance between the factor u_1 and the inverse function $\Delta^{-1}(u_2)$ for which it works. Then, we have to check successively for the values $d_{min} = 0, 1 \dots$ until a tiling is obtained (if it exists).

Smooth 2 $\times$ 2 patterns

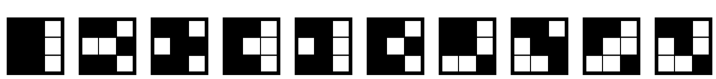
Consider first, the 2×2 patterns. Using smooth factors of length 2, and the method described above, it can be checked that all the 2×2 square patterns occur in some tiling. More precisely, the index of first occurrence for the 16 square patterns is shown in the following table where the first pattern exists for instance in the Kolakoski tiling.



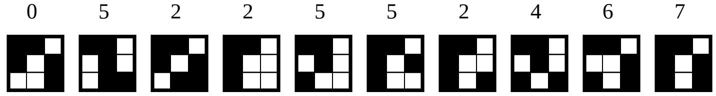
0 2 2 0 0 2 2 0 2 0 0 1 2 0 0 1

Smooth 3 $\times$ 3 patterns

In the same way we determine the index I_{min} of such patterns in a tiling.All the 6^3 patterns are smooth and due to this higher number we list only some of them.



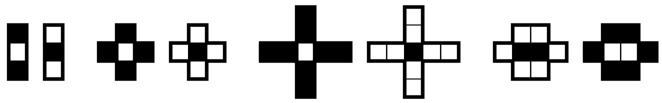
0 5 2 2 5 5 2 4 6 7



0 11 3 11 4 4 3 4 5 14

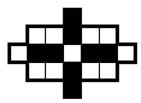
Palindromic patterns

In the case of two-dimensional palindromic patterns obtained by the stacking of palindromic smooth words, we obtained the following results for small patterns

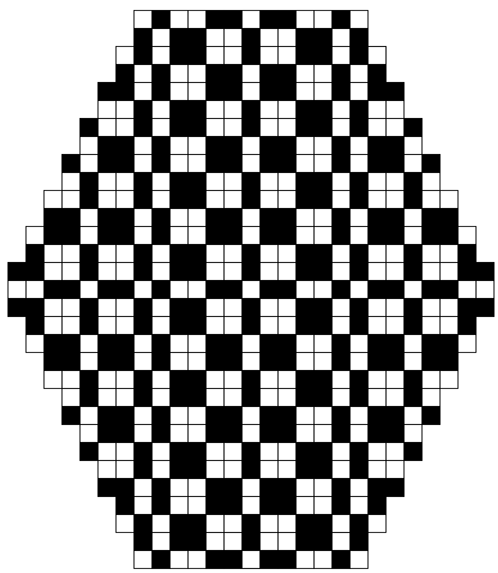


0 0 1 0 0 2 1 4

Moreover an extensive search based on various tilings, shows that these small smooth palindromic patterns occur frequently. On the other hand, whether the following pattern is smooth or not is an open problem:



If not, then the following large palindromic pattern would not occur in any tiling.



Some open problems

One basic problem is to prove that there exist smooth patterns which do not occur in any smooth tiling and to exhibit such patterns. We observed that all the 2×2 and 3×3 smooth patterns seem to appear with about the same frequencies. Is this true for $n \times n$ smooth patterns? What is the frequency of appearence of a given pattern in a smooth tiling? Another problem is to estimate or find an exact value of the maximum distance between two consecutive occurences of a given pattern in smooth tilings.

5.2 Periodic Tilings

We consider now periodic tilings, those corresponding to periodic words of Σ^*. From here, we restrict our study to quarter plane tiling constructed from $u \in \Sigma^*$ and defined by

$$T_u[i,j] = w_i[j] \quad \forall i, 1 \le i \le |u| \quad \text{and} \quad \forall j, 1 \le j \le |w_i|,$$

where

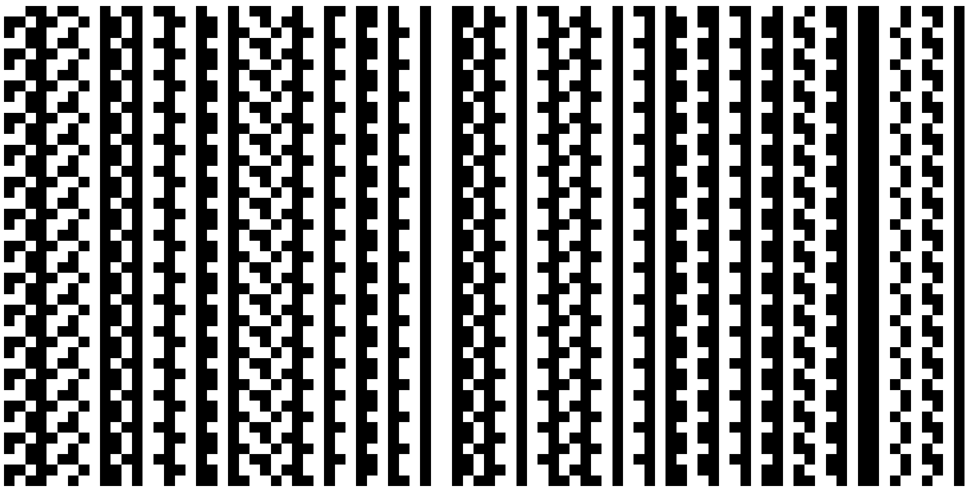
$$w_n = \begin{cases} u[|u|], & \text{if } n = |u|; \\ \Delta_{u[n]}^{-1}(w_{n+1}) & \text{if } 1 \leq n < |u|. \end{cases}$$

This tiling can be extended to bi-infinite words and then correspond to the positive half plane $\mathbb{Z}^+ \times \mathbb{Z}$.

Example. The periodic word

$$(122)^\omega = 122122122122\cdots$$

yields the periodic tiling



Let T_u be the tiling corresponding $u = (p)^\omega \in \Sigma^\omega$. The *strip* of T_u is the infinite tile determined by the $|p|$ consecutive lines

$$L_1, L_2, L_3, \ldots, L_{|p|}$$

of T_u. Obviously, the strip of T_u is of width $|p|$.

Example. The strip of the periodic word $(122)^\omega = 122122122122\cdots$ is

1	1	2	2	1	2	2	1	1	2	1	2	2	1	2	2	1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2
2	2	1	2	2	1	1	2	1	2	2	1	2	1	1	2	2	1	2	2	1	2	1	1	2	1	2	2	1	1	2	1
2	1	2	2	1	1	2	1	1	2	2	1	2	1	1	2	1	1	2	2	1	2	2	1	1	2	1	2	2	1	2	1

One can see that a periodic tiling T_u can be constructed by translating vertically its strip. In symbol, if S_{T_u} is the strip of T_u and $t = (0, |p|)$ is a vertical translation of $|p|$ unities, then

$$T_u = \bigcup_{i \in \mathbb{N}} S_{T_u} t^i.$$

From this last construction, it is clear that each column of the tiling T_u is a periodic word of period $|p|$ and we have the following properties.

Observation 1 *Let $u = (p)^\omega \in \Sigma^\omega$ of minimal period p and T_u the associated tiling. Then*

(i) *there are $2^{|p|}$ different columns in T_u;*

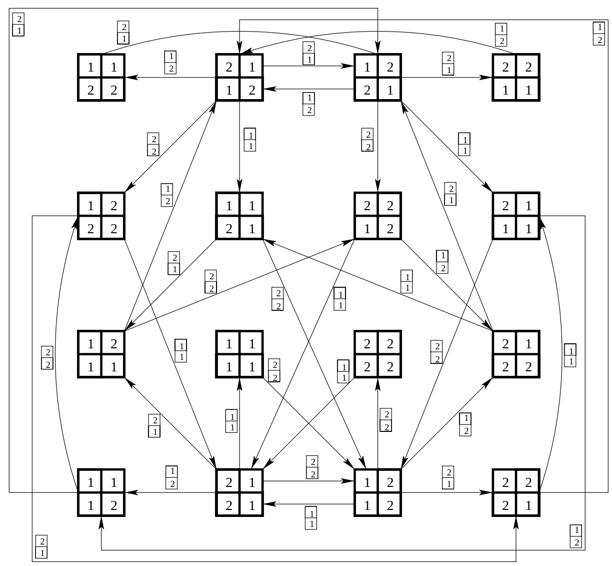
(ii) *each column of T_u is recurrent;*

In order to prove the first observation, a $|p|$ dimensional de Bruijn graph can be associated to each periodic tiling considering the strip of the tiling as a $|p|$ dimensional word.

Example. The periodic word $(12)^\omega = 1212121212\cdots$ yields a tiling of strip

1	1	2	1	1	2	2	1	2	2	1	2	1	1	2	2	1	2	2	1	1	2	1	2	2	1	2	1	1	2
2	1	2	2	1	2	1	1	2	2	1	2	2	1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	2	1

where each word of length 2 over $\Sigma = \{1, 2\}$ appear as a column word. Its de Bruijn graph is

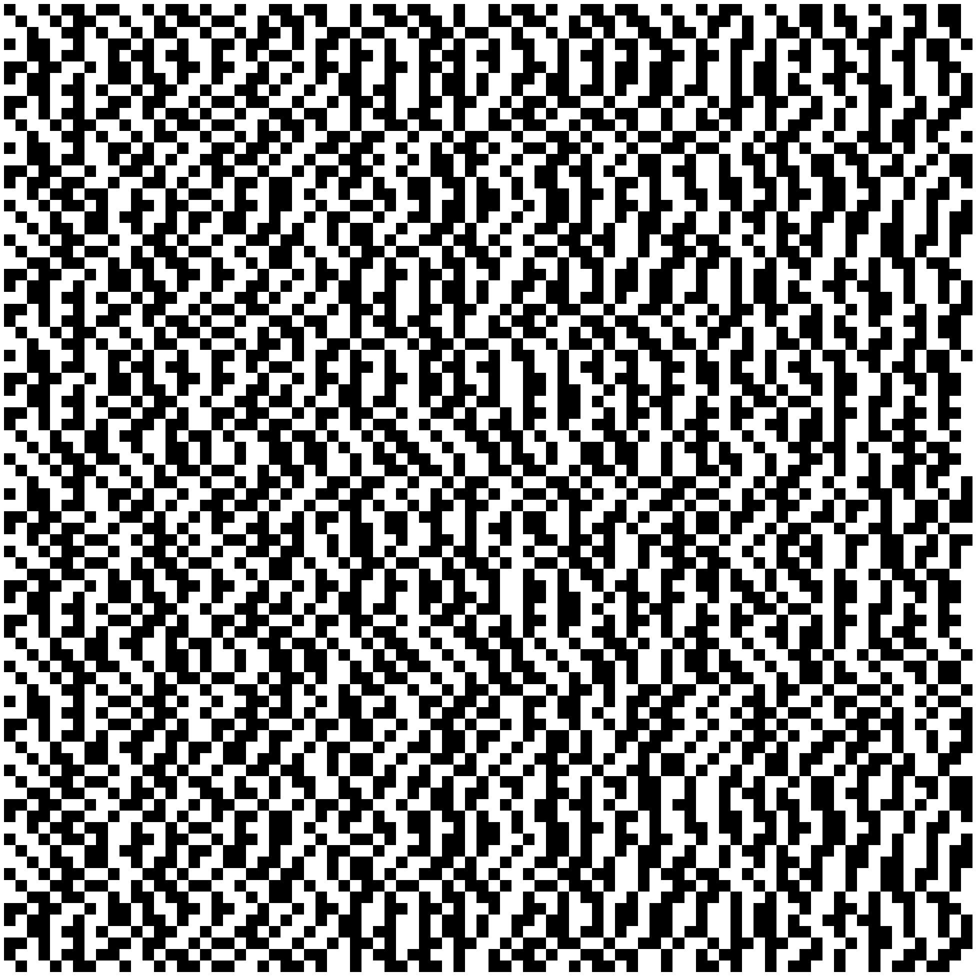


5.3 FixPoint tilings

The fix-points for Φ (see [2, 3]) are $\mathrm{Fix}(\Phi) = \{X \in \mathcal{K} \mid \Phi(X) = X\}$:

$$\{211212212211212212112\cdots,\ 212211211221221211211\cdots,\ 221221121221211221221\cdots\}$$

The tiling associated to the first one is



At first glance, the tilings seem to be chaotic, however the distribution of the Kolakoski word's prefixes in the fix-point tilings shows a surprising synchronicity. The following tables list the positions of the two longest Kolakoski prefixes, found in tilings of size 1000×1000, according to their length. The couple (i, j) refer to the i^{th} colum of the j^{th} line.

Prefixe	Tilings		
	$2112\cdots$	$2122\cdots$	$2212\cdots$
$K[1..149]$	(73,158) (73,303) (73,527) (73,707) (474,93) (474,138) (474,258) (474,364) (474,444) (474,617) (474,662) (474,768)	(73,54) (73,199) (73,595) (73,626) (73,715) (474,9) (474,154) (474,259) (474,304) (474,424) (474,530) (474,575) (474,695) (474,788) (474,967)	(73,197) (73,320) (73,347) (73,467) (73,806) (73,895) (474,33) (474,420)

Prefixe	Tilings		
	2112···	2122···	2212···
$K[1..255]$	(712,92) (712,137) (712,257) (712,363) (712,443) (712,616) (712,661) (712,767)	(712,8) (712,153) (712,258) (712,303) (712,423) (712,529) (712,574) (712,694) (712,787) (712,966)	(712,32) (712,419)

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